1

Vectors, Tensors, and Equations of Elasticity

1.1 Prove the following properties of δ_{ij} and ε_{ijk} (assume $i, j = 1, 2, 3$ when they are dummy indices):

(a) $F_{ij}\delta_{jk} = F_{ik}$

(b) $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

(c) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

(d) $\varepsilon_{ijk}F_{ij} = 0$ whenever $F_{ij} = F_{ji}$ (symmetric)

Solution:

1.1(a) Expanding the expression

$$F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$$

Of the three terms on the right hand side, only one is nonzero. It is equal to F_{i1} if $k = 1$, F_{i2} if $k = 2$, or F_{i3} if $k = 3$. Thus, it is simply equal to F_{ik}.

1.1(b) By actual expansion, we have

$$\begin{aligned}\delta_{ij}\delta_{ij} &= \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3}\\ &= (\delta_{11}\delta_{11} + 0 + 0) + (0 + \delta_{22}\delta_{22} + 0) + (0 + 0 + \delta_{33}\delta_{33})\\ &= 3\end{aligned}$$

and

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Alternatively, using $F_{ij} = \delta_{ij}$ in Problem 1.1a, we have $\delta_{ij}\delta_{jk} = \delta_{ik}$, where i and k are free indices that can any value. In particular, for $i = k$, we have the required result.

1.1(c) Using the ε-δ identity and the result of Problem 1.1(b), we obtain

$$\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ij} = 9 - 3 = 6$$

1.1(d) We have

$$\begin{aligned} F_{ij}\varepsilon_{ijk} &= -F_{ij}\varepsilon_{jik} \quad \text{(interchanged } i \text{ and } j) \\ &= -F_{ji}\varepsilon_{ijk} \quad \text{(renamed } i \text{ as } j \text{ and } j \text{ as } i) \end{aligned}$$

Since $F_{ji} = F_{ij}$, we have

$$\begin{aligned} 0 &= (F_{ij} + F_{ji})\,\varepsilon_{ijk} \\ &= 2F_{ij}\varepsilon_{ijk} \end{aligned}$$

The converse also holds, i.e., if $F_{ij}\varepsilon_{ijk} = 0$, then $F_{ij} = F_{ji}$. We have

$$\begin{aligned} 0 &= F_{ij}\varepsilon_{ijk} \\ &= \frac{1}{2}\left(F_{ij}\varepsilon_{ijk} + F_{ij}\varepsilon_{ijk}\right) \\ &= \frac{1}{2}\left(F_{ij}\varepsilon_{ijk} - F_{ij}\varepsilon_{jik}\right) \quad \text{(interchanged } i \text{ and } j) \\ &= \frac{1}{2}\left(F_{ij}\varepsilon_{ijk} - F_{ji}\varepsilon_{ijk}\right) \quad \text{(renamed } i \text{ as } j \text{ and } j \text{ as } i) \\ &= \frac{1}{2}\left(F_{ij} - F_{ji}\right)\varepsilon_{ijk} \end{aligned}$$

from which it follows that $F_{ji} = F_{ij}$.

♠ **New Problem 1.1:** Show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

Solution: Write the position vector in cartesian component form using the index notation

$$\mathbf{r} = x_j \hat{\mathbf{e}}_j \tag{1}$$

Then the square of the magnitude of the position vector is

$$\begin{aligned} r^2 = \mathbf{r}\cdot\mathbf{r} &= (x_i\hat{\mathbf{e}}_i)\cdot(x_j\hat{\mathbf{e}}_j) = x_i x_j \delta_{ij} \\ &= x_i x_i = x_k x_k \end{aligned} \tag{2}$$

Its derivative of r with respect to x_i can be obtained from

$$\begin{aligned} \frac{\partial r^2}{\partial x_i} &= \frac{\partial}{\partial x_i}(x_k x_k) \\ &= \frac{\partial x_k}{\partial x_i} x_k + x_k \frac{\partial x_k}{\partial x_i} \\ &= 2\frac{\partial x_k}{\partial x_i} x_k = 2\delta_{ik} x_k = 2x_i \end{aligned}$$

Hence

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \tag{3}$$

1.2 Let $\mathbf{r}$ denote a position vector. Show that:

(a) grad $(r^n) = nr^{n-2}\mathbf{r}$

(b) $\nabla^2(r^n) = n(n+1)r^{n-2}$

(c) div $(\mathbf{r}) = 3$

(d) curl $(\mathbf{r}f(r)) = \mathbf{0}$, where $f(r)$ is an arbitrary continuous function of r with continuous first derivatives

Solution:

1.2(a) We have

$$\nabla(r^n) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(r^n) = nr^{n-1}\hat{\mathbf{e}}_i \frac{\partial r}{\partial x_i} = nr^{n-2}x_i\hat{\mathbf{e}}_i = nr^{n-2}\mathbf{r}$$

where the result from Eq. (3) of Problem 1.1 is used in arriving at the last step.

1.2(b) From the result of the above exercise, we have

$$\begin{aligned}\nabla^2(r^n) &= (\nabla\cdot\nabla)(r^n) = \nabla\cdot[\nabla(r^n)]\\ &= \left(\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}\right)\cdot\left(nr^{n-2}x_i\hat{\mathbf{e}}_i\right) = n(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_i)\frac{\partial}{\partial x_j}\left(r^{n-2}x_i\right)\\ &= n\delta_{ij}\left[(n-2)r^{n-3}\frac{x_j}{r}x_i + r^{n-2}\delta_{ij}\right]\\ &= n\left[(n-2)r^{n-2} + 3r^{n-2}\right] = n(n+1)r^{n-2}\end{aligned}$$

1.2(c) Using Eq. (3) of Problem 1.1(b), we obtain

$$\nabla\cdot\mathbf{r} = \left(\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}\right)\cdot(x_i\hat{\mathbf{e}}_i) = (\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_i)\frac{\partial x_i}{\partial x_j} = \delta_{ij}\delta_{ij} = 3$$

1.2(d) We obtain

$$\begin{aligned}\text{culr}(f\mathbf{r}) &= \nabla\times[f\mathbf{r}]\\ &= \left(\hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}\right)\times(f(r)x_i\hat{\mathbf{e}}_i) = (\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_i)\frac{\partial}{\partial x_j}[f(r)x_i]\\ &= \varepsilon_{jik}\hat{\mathbf{e}}_k\left[f'(r)\frac{\partial r}{x_j}x_i + f(r)\delta_{ij}\right]\\ &= \varepsilon_{jik}\hat{\mathbf{e}}_k\left[\frac{f'(r)}{r}x_jx_i + f(r)\delta_{ij}\right]\end{aligned}$$

The two terms in the square brackets, x_ix_j and δ_{ij} are symmetric, hence, by Problem 1.1(d) the expression in the last line is zero.

♠ **New Problem 1.2:** Let $[A]$ and $[B]$ be $m \times n$ and $n \times p$ matrices, respectively. Show that

$$([A][B])^{\mathrm{T}} = [B]^{\mathrm{T}}[A]^{\mathrm{T}} \tag{1}$$

Solution: By definition of the product of two matrices, we have $[A][B] = [C]$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Then the transpose of $[C]$ has the coefficients

$$\begin{aligned} c_{ji} &= \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk} \\ &= \sum_{k=1}^{n} (b_{ik})^{\mathrm{T}} (a_{kj})^{\mathrm{T}} \end{aligned}$$

which implies the result in Eq. (1).

1.3 If $[B]$ is a symmetric $n \times n$ matrix and $[C]$ is any $n \times n$ matrix, show that $[C]^{\mathrm{T}}[B][C]$ is symmetric.

Solution: Let $[A] = [B][C]$. Using Eq. (1) of New Problem 1.2, we have

$$\left([C]^{\mathrm{T}}[A]\right)^{\mathrm{T}} = [A]^{\mathrm{T}}[C] = [C]^{\mathrm{T}}[B][C]$$

where we have also used the identity

$$\left([C]^{\mathrm{T}}\right)^{\mathrm{T}} = [C]$$

♠ **New Problem 1.3:** Show that the dot and cross can be interchanged without changing the value in the *scalar triple product*

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \tag{1}$$

Solution: We have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} =& A_i \hat{\mathbf{e}}_i \cdot B_j C_k \varepsilon_{jkm} \hat{\mathbf{e}}_m = A_i B_j C_k \varepsilon_{jkm} \delta_{im} \\ =& A_i B_j C_k \varepsilon_{jki} = A_i B_j C_k \varepsilon_{ijk} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \end{aligned}$$

Since i, j, and k can be permuted in a cyclic order, it also follows that

$$A_i B_j C_k \varepsilon_{ijk} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$$

and $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A}$.

1.4 Establish the ε-δ identity of Eq. (1.2.15).

Solution: The ε-δ identity follows directly from the vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{1}$$

by letting

$$\mathbf{A} = \hat{\mathbf{e}}_i, \quad \mathbf{B} = \hat{\mathbf{e}}_j, \quad \mathbf{C} = \hat{\mathbf{e}}_m, \quad \mathbf{A} = \hat{\mathbf{e}}_n$$

We obtain

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot (\hat{\mathbf{e}}_m \times \hat{\mathbf{e}}_n) = (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_m)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_n) - (\hat{\mathbf{e}}_i \cdot \mathbf{e}_n)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m)$$

$$\varepsilon_{ijk}\hat{\mathbf{e}}_k \cdot \varepsilon_{mnp}\hat{\mathbf{e}}_p = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

or

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

which was to be proved. Note that $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$.

1.5 Prove that the determinant of a 3×3 matrix $[C]$ can be expressed in the form

$$|C| = \varepsilon_{ijk}\, c_{1i}\, c_{2j}\, c_{3k} \tag{a}$$

and, thus, prove

$$|C| = \frac{1}{6}\varepsilon_{ijk}\, \varepsilon_{rst}\, c_{ir}\, c_{js}\, c_{kt} \tag{b}$$

where c_{ij} is the element occupying the ith row and the jth column of $[C]$.

Solution: First we note the definition of the cross product of two vectors

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{3}$$

and the "scalar triple product" of vectors

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{4}$$

Now let

$$\mathbf{A} = c_{1i}\hat{\mathbf{e}}_i \equiv \mathbf{C}_1, \quad \mathbf{B} = c_{2j}\hat{\mathbf{e}}_j \equiv \mathbf{C}_2, \quad \mathbf{C} = c_{3k}\hat{\mathbf{e}}_k \equiv \mathbf{C}_3$$

in Eq. (3). We obtain

$$\begin{aligned} \mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3) &= c_{1i}\hat{\mathbf{e}}_i \cdot (c_{2j}\hat{\mathbf{e}}_j \times c_{3k}\hat{\mathbf{e}}_k) \\ &= \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \equiv |C| \end{aligned}$$

or

$$\begin{aligned}|C| &= c_{1i}\hat{\mathbf{e}}_i \cdot (c_{2j}\hat{\mathbf{e}}_j \times c_{3k}\hat{\mathbf{e}}_k)\\ &= c_{1i}c_{2j}c_{3k}\varepsilon_{ijk}\end{aligned}$$

which is the same as Eq. (1). Next consider the product $\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t)$:

$$\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t) = c_{ri}c_{sj}c_{tk}\varepsilon_{ijk}$$

Multiplying both sides with ε_{rst} and expanding, we arrive at

$$\begin{aligned}c_{ri}c_{sj}c_{tk}\varepsilon_{rst}\varepsilon_{ijk} &= \varepsilon_{rst}[\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t)]\\ &= \varepsilon_{1st}[\mathbf{C}_1 \cdot (\mathbf{C}_s \times \mathbf{C}_t)] + \varepsilon_{2st}[\mathbf{C}_2 \cdot (\mathbf{C}_s \times \mathbf{C}_t)]\\ &\quad + \varepsilon_{3st}[\mathbf{C}_3 \cdot (\mathbf{C}_s \times \mathbf{C}_t)]\\ &= \mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3) - \mathbf{C}_1 \cdot (\mathbf{C}_3 \times \mathbf{C}_2)\\ &\quad + \mathbf{C}_2 \cdot (\mathbf{C}_3 \times \mathbf{C}_1) - \mathbf{C}_2 \cdot (\mathbf{C}_1 \times \mathbf{C}_3)\\ &\quad + \mathbf{C}_3 \cdot (\mathbf{C}_1 \times \mathbf{C}_2) - \mathbf{C}_3 \cdot (\mathbf{C}_2 \times \mathbf{C}_1)\\ &= 6[\mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3)] = 6|C|\end{aligned}$$

where we have used the identity in Eq. (1) of New Problem 1.3.

1.6 Using Cramer's rule determine the solution to the following equations:

(a)

$$\begin{aligned}2x_1 - x_2 &= 1\\ -x_1 + 2x_2 - x_3 &= 2\\ -x_2 + 2x_3 &= 2\end{aligned}$$

(b)

$$\frac{2b}{h^3}\begin{bmatrix}12 & 0 & 3h\\ 0 & 4h^2 & h^2\\ 3h & h^2 & 2h^2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{f_0 h}{12}\begin{Bmatrix}12\\ 0\\ h\end{Bmatrix}$$

where b, f_0, and h are constants

Solution:

1.6(a) The matrix form of the equations is

$$\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}1\\ 2\\ 2\end{Bmatrix}$$

Using Cramer's rule we obtain

$$x_1 = \frac{1}{|A|}\begin{vmatrix}1 & -1 & 0\\ 2 & 2 & -1\\ 2 & -1 & 2\end{vmatrix} = \frac{1}{|A|}[(4-1) + (4+2) - 0] = \frac{9}{|A|}$$

$$x_2 = \frac{1}{|A|}\begin{vmatrix}2 & 1 & 0\\ -1 & 2 & -1\\ 0 & 2 & 2\end{vmatrix} = \frac{1}{|A|}[2(4+2) - (-2-0) - 0] = \frac{14}{|A|}$$

$$x_3 = \frac{1}{|A|}\begin{vmatrix}2 & -1 & 1\\ -1 & 2 & 2\\ 0 & -1 & 2\end{vmatrix} = \frac{1}{|A|}[2(4+2) + (-2+1) + 0] = \frac{11}{|A|}$$

where the determinant $|A|$ of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(4-1) + (-2-0) - 0 = 4$$

Hence, $x_1 = 9/4,\ x_2 = 14/4$, and $x_3 = 11/4$.

1.6(b) We have

$$\begin{bmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ 3h & h^2 & 2h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{f_0 h^4}{24b} \begin{Bmatrix} 12 \\ 0 \\ h \end{Bmatrix}$$

The determinant of the coefficient matrix is $|A| = 12(8h^4 - h^4) + 3h(-12h^3) = 48h^4$. Using Cramer's rule we obtain

$$x_1 = \frac{\alpha}{|A|} \begin{vmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ h & h^2 & 2h^2 \end{vmatrix} = \frac{\alpha}{|A|}[12(8h^4 - h^4) + h(-12h^3)] = \frac{72\alpha}{|A|}$$

$$x_2 = \frac{\alpha}{|A|} \begin{vmatrix} 12 & 12 & 3h \\ 0 & 0 & h^2 \\ 3h & h & 2h^2 \end{vmatrix} = \frac{\alpha}{|A|}[12(-h^3) + 3h(12h^2)] = \frac{24\alpha}{|A|}$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} 12 & 0 & 12 \\ 0 & 4h^2 & 0 \\ 3h & h^2 & h \end{vmatrix} = \frac{\alpha}{|A|}[12(4h^3) + 3h(-48h^2)] = -\frac{96\alpha}{|A|}$$

Hence, $x_1 = 3\alpha/2,\ x_2 = \alpha/(2h)$, and $x_3 = -2\alpha/h$, where $\alpha = (f_0 h^4/24b)$.

1.7 Let [C] be a 3×3 matrix, [I] be a 3×3 identity matrix, and λ be a scalar. Show that

$$\det[C - \lambda I] = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3$$

where

$$I_1 = c_{ii},\ I_2 = \frac{1}{2}(c_{ii}c_{jj} - c_{ij}c_{ij}),\ I_3 = |C|$$

Solution: Using the result of Problem 1.5(b) and the ε-δ identity, we obtain

$$\begin{aligned}
|C - \lambda I| &= \frac{1}{6}\varepsilon_{ijk}\varepsilon_{rst}(c_{ir} - \lambda\delta_{ir})(c_{js} - \lambda\delta_{js})(c_{kt} - \lambda\delta_{kt}) \\
&= \frac{1}{6}\varepsilon_{ijk}\varepsilon_{rst}\left[-\lambda^3\delta_{ir}\delta_{js}\delta_{kt} + \lambda^2\left(c_{ir}\delta_{js}\delta_{kt} + c_{kt}\delta_{ir}\delta_{js} + c_{js}\delta_{ir}\delta_{kt}\right)\right. \\
&\quad \left. - \lambda\left(c_{ir}c_{js}\delta_{kt} + c_{ir}\delta_{js}c_{kt} + \delta_{ir}c_{js}c_{kt}\right) + c_{ir}c_{js}c_{kt}\ \right] \\
&= -\lambda^3 + \frac{\lambda^2}{6}\left(c_{ir}\varepsilon_{ijk}\varepsilon_{rjk} + c_{kt}\varepsilon_{ijk}\varepsilon_{ijt} + \varepsilon_{ijk}\varepsilon_{isk}c_{js}\right) \\
&\quad + \frac{\lambda}{6}\left(\varepsilon_{ijk}\varepsilon_{rsk}c_{ir}c_{js} + \varepsilon_{ijk}\varepsilon_{rjt}c_{ir}c_{kt} + \varepsilon_{ijk}\varepsilon_{ist}c_{js}c_{kt}\right) \\
&\quad + \frac{1}{6}\varepsilon_{ijk}\varepsilon_{rst}c_{ir}c_{js}c_{kt} \\
&= -\lambda^3 + c_{ii}\lambda^2 + \frac{\lambda}{2}\left(c_{ii}c_{jj} - c_{ij}c_{ji}\right) + |C|
\end{aligned}$$

1.8 If we identify a second-order tensor $\mathbf{A}$ associated with the direction cosines $a_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j$ [see Eq. (1.2.57)]

$$\mathbf{A} = a_{ij}\, \hat{\bar{\mathbf{e}}}_i \hat{\mathbf{e}}_j$$

show that (a) $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}$, (b) $\mathbf{A} \cdot \mathbf{A}^{\mathrm{T}} = \mathbf{I}$, and (c) $\mathbf{A} : \mathbf{A} = 3$.

Solution:

1.8(a) We have

$$\mathbf{A} \cdot \mathbf{A} = a_{ij} a_{kp} a_{kj} \hat{\bar{\mathbf{e}}}_i \hat{\mathbf{e}}_p = a_{ip} \hat{\bar{\mathbf{e}}}_i \hat{\mathbf{e}}_p = \mathbf{A}$$

where we have used the identity in Eq. (1.2.61).

1.8(b) We have

$$\mathbf{A} \cdot \mathbf{A}^{\mathrm{T}} = a_{ij} a_{kp} \delta_{jp} \hat{\bar{\mathbf{e}}}_i \hat{\bar{\mathbf{e}}}_k = a_{ij} a_{kj} \hat{\bar{\mathbf{e}}}_i \hat{\bar{\mathbf{e}}}_k = \delta_{ip} \hat{\bar{\mathbf{e}}}_i \hat{\bar{\mathbf{e}}}_k = \mathbf{I}$$

where we have used the identity in Eq. (1.2.61).

1.8(c) We have

$$\begin{aligned}\mathbf{A} : \mathbf{A} &= a_{ij} a_{mn} \left(\hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_n\right) \left(\hat{\bar{\mathbf{e}}}_j \cdot \hat{\mathbf{e}}_m\right) \\ &= a_{ij} a_{mn} a_{in} a_{mj} = \delta_{jn} \delta_{nj} = \delta_{jj} = 3\end{aligned}$$

where we have used the identity in Eq. (1.2.61) repeatedly.

1.9 Use the definition $\nabla^2 = \nabla \cdot \nabla$ to show that the Laplacian operator in the cylindrical coordinate system is given by

$$\nabla^2 = \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\Big]$$

Solution: Using the definition (1.2.30) of ∇ and the derivatives of the basis vectors (1.2.29)

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\,, \quad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$

we obtain

$$\begin{aligned}\nabla \cdot \nabla &= \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \\ &= \hat{\mathbf{e}}_r \cdot \frac{\partial}{\partial r}\left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \\ &\quad + \frac{1}{r}\hat{\mathbf{e}}_\theta \cdot \frac{\partial}{\partial \theta}\left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \\ &\quad + \hat{\mathbf{e}}_z \cdot \frac{\partial}{\partial z}\left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\hat{\mathbf{e}}_\theta \cdot \left(\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial^2}{\partial \theta^2}\right) + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\end{aligned}$$

1.10 Show that the gradient of a vector $\mathbf{u}$ in the cylindrical coordinate system is given by

$$\begin{aligned}
\nabla\mathbf{u} = &\frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z \\
&+ \frac{1}{r}\frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z \\
&+ \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r
\end{aligned}$$

Solution: We have

$$\begin{aligned}
\nabla\mathbf{u} &= \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)(u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\mathbf{e}_z) \\
&= \hat{\mathbf{e}}_r\frac{\partial}{\partial r}(u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\mathbf{e}_z) \\
&\quad + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta}(u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\mathbf{e}_z) \\
&\quad + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}(u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\mathbf{e}_z) \\
&= \hat{\mathbf{e}}_r\left(\frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z\right) \\
&\quad + \frac{\hat{\mathbf{e}}_\theta}{r}\left(\frac{\partial u_r}{\partial \theta}\hat{\mathbf{e}}_r + u_r\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\hat{\mathbf{e}}_\theta + u_\theta\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_z\right) \\
&\quad + \hat{\mathbf{e}}_z\left(\frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\right) \\
&= \hat{\mathbf{e}}_r\left(\frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z\right) \\
&\quad + \frac{\hat{\mathbf{e}}_\theta}{r}\left(\frac{\partial u_r}{\partial \theta}\hat{\mathbf{e}}_r + u_r\hat{\mathbf{e}}_\theta + \frac{\partial u_\theta}{\partial \theta}\hat{\mathbf{e}}_\theta - u_\theta\hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_z\right) \\
&\quad + \hat{\mathbf{e}}_z\left(\frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\right)
\end{aligned}$$

1.11 Show that the curl of a vector $\mathbf{u}$ in the cylindrical coordinate system is given by

$$\begin{aligned}
\nabla\times\mathbf{u} = &\hat{\mathbf{e}}_r\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) + \hat{\mathbf{e}}_\theta\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \\
&+ \hat{\mathbf{e}}_z\left(\frac{\partial u_\theta}{\partial r} + \frac{u\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)
\end{aligned}$$

Solution: We have

$$\nabla\times\mathbf{u} = \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)\times(u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\mathbf{e}_z)$$

$$
\begin{aligned}
&= \hat{\mathbf{e}}_r \times \frac{\partial}{\partial r}\left(u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\hat{\mathbf{e}}_\theta \times \frac{\partial}{\partial \theta}\left(u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z\right) \\
&\quad + \hat{\mathbf{e}}_z \times \frac{\partial}{\partial z}\left(u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z\right) \\
&= \hat{\mathbf{e}}_r \times \left(\frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\hat{\mathbf{e}}_\theta \times \left(\frac{\partial u_r}{\partial \theta}\hat{\mathbf{e}}_r + u_r \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\hat{\mathbf{e}}_\theta + u_\theta \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_z\right) \\
&\quad + \hat{\mathbf{e}}_z \times \left(\frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\right) \\
&= \hat{\mathbf{e}}_r \times \left(\frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\hat{\mathbf{e}}_\theta \times \left(\frac{\partial u_r}{\partial \theta}\hat{\mathbf{e}}_r + u_r \hat{\mathbf{e}}_\theta + \frac{\partial u_\theta}{\partial \theta}\hat{\mathbf{e}}_\theta - u_\theta \hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_z\right) \\
&\quad + \hat{\mathbf{e}}_z \times \left(\frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\right) \\
&= \frac{\partial u_\theta}{\partial r}\left(\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta\right) + \frac{\partial u_z}{\partial r}\left(\hat{\mathbf{e}}_r \times \mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\left[\frac{\partial u_r}{\partial \theta}\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r\right) - u_\theta\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r\right) + \frac{\partial u_z}{\partial \theta}\left(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z\right)\right] \\
&\quad + \frac{\partial u_r}{\partial z}\left(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r\right) + \frac{\partial u_\theta}{\partial z}\left(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\theta\right) \\
&= \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_z - \frac{\partial u_z}{\partial r}\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(-\frac{\partial u_r}{\partial \theta}\hat{\mathbf{e}}_z + u_\theta \hat{\mathbf{e}}_z + \frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_r\right) \\
&\quad + \frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_\theta - \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_r
\end{aligned}
$$

♠ **New Problem 1.4:** Find the divergence of a vector in the cylindrical coordinate system.

Solution: We have

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z\right) \\
&= \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r \frac{\partial u_r}{\partial r} + \hat{\mathbf{e}}_\theta \cdot \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\frac{u_r}{r} + \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z \frac{\partial u_z}{\partial z} \\
&= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\end{aligned}
$$

1.12 For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \cdot \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \cdot \mathbf{S} = &\left[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r}\left(S_{rr} - S_{\theta\theta}\right)\right]\hat{\mathbf{e}}_r \\ &+ \left[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r}\left(S_{r\theta} + S_{\theta r}\right)\right]\hat{\mathbf{e}}_\theta \\ &+ \left[\frac{\partial S_{rz}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r}S_{rz}\right]\hat{\mathbf{e}}_z\end{aligned}$$

Solution: Using the del operator in the cylindrical coordinate system, the divergence of the tensor $\mathbf{S}$ is computed as

$$\begin{aligned}&\left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)\cdot\left[S_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + S_{r\theta}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + S_{\theta r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \cdots + S_{zz}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z\right] \\ &\quad = \frac{\partial S_{rr}}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial S_{r\theta}}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial S_{rz}}{\partial r}\hat{\mathbf{e}}_z + \frac{1}{r}\Big[S_{rr}\hat{\mathbf{e}}_\theta\cdot\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\hat{\mathbf{e}}_r + S_{r\theta}\hat{\mathbf{e}}_\theta\cdot\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\hat{\mathbf{e}}_\theta \\ &\qquad + \frac{\partial S_{\theta\theta}}{\partial \theta}\hat{\mathbf{e}}_\theta + S_{\theta\theta}\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial S_{\theta r}}{\partial \theta}\hat{\mathbf{e}}_r + S_{\theta r}\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + S_{zr}\hat{\mathbf{e}}_\theta\cdot\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\hat{\mathbf{e}}_z + \frac{\partial S_{\theta z}}{\partial \theta}\hat{\mathbf{e}}_z\Big] \\ &\qquad + \frac{\partial S_{zr}}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial S_{z\theta}}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial S_{zz}}{\partial z}\hat{\mathbf{e}}_z \\ &\quad = \frac{\partial S_{rr}}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial S_{r\theta}}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\partial S_{rz}}{\partial r}\hat{\mathbf{e}}_z + \frac{S_{rr}}{r}\hat{\mathbf{e}}_r + \frac{S_{r\theta}}{r}\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta}\hat{\mathbf{e}}_\theta - \frac{S_{\theta\theta}}{r}\hat{\mathbf{e}}_r \\ &\qquad + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta}\hat{\mathbf{e}}_r + \frac{S_{\theta r}}{r}\hat{\mathbf{e}}_\theta + \frac{S_{rz}}{r}\hat{\mathbf{e}}_z + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta}\hat{\mathbf{e}}_z + \frac{\partial S_{zr}}{\partial z}\hat{\mathbf{e}}_r + \frac{\partial S_{z\theta}}{\partial z}\hat{\mathbf{e}}_\theta + \frac{\partial S_{zz}}{\partial z}\hat{\mathbf{e}}_z,\end{aligned}$$

where the following derivatives of the base vectors are accounted for:

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r, \quad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta.$$

Collecting the coefficients of $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_z$, we obtain the required result.

1.13 For an arbitrary second-order tensor $\mathbf{S}$ show that $\nabla \times \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \times \mathbf{S} = &\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r\left(\frac{1}{r}\frac{\partial S_{zr}}{\partial \theta} - \frac{\partial S_{\theta r}}{\partial z} - \frac{1}{r}S_{z\theta}\right) + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta\left(\frac{\partial S_{r\theta}}{\partial z} - \frac{\partial S_{z\theta}}{\partial r}\right) + \\ &\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z\left(\frac{1}{r}S_{\theta z} - \frac{1}{r}\frac{\partial S_{rz}}{\partial \theta} + \frac{\partial S_{\theta z}}{\partial r}\right) + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta\left(\frac{1}{r}\frac{\partial S_{z\theta}}{\partial \theta} - \frac{\partial S_{\theta\theta}}{\partial z} + \frac{1}{r}S_{zr}\right) + \\ &\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r\left(\frac{\partial S_{rr}}{\partial z} - \frac{\partial S_{zr}}{\partial r}\right) + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_z\left(\frac{1}{r}\frac{\partial S_{zz}}{\partial \theta} - \frac{\partial S_{\theta z}}{\partial z}\right) + \\ &\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r\left(\frac{\partial S_{\theta r}}{\partial r} - \frac{1}{r}\frac{\partial S_{rr}}{\partial \theta} + \frac{1}{r}S_{r\theta} + \frac{1}{r}S_{\theta r}\right) + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z\left(\frac{\partial S_{rz}}{\partial z} - \frac{\partial S_{zz}}{\partial r}\right) + \\ &\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta\left(\frac{\partial S_{\theta\theta}}{\partial r} + \frac{1}{r}S_{\theta\theta} - \frac{1}{r}S_{rr} - \frac{1}{r}\frac{\partial S_{r\theta}}{\partial \theta}\right)\end{aligned}$$