Third Edition

RADIATIVE HEAT TRANSFER

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SOLUTION MANUAL

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PREFACE

This manual/web page contains the solutions to many (but not all) of the problems that are given at the end of each chapter, in particular for problems on topics that are commonly covered in a first (or, at least, second) graduate course on radiative heat transfer. Thus, solutions to problems of Chapters 1 through 6, 9 through 11, 13, 14 and 18 are almost complete; for other chapters (7, 15, 16, 19) only around half of solutions are given, for problems on the more basic aspects covered in that chapter. Quite a few solutions, together with Fortran90 codes, are also given for Chapter 20, not so much intended as homework solutions, but perhaps useful for semester projects and/or to aid the researcher to write his/her first Monte Carlo code. At this date no problem solutions are provided for the remaining chapters. However, it is the intent to periodically update this web-based solution manual, at which time additional solutions may be posted.

Most solutions have been checked, and some have been rechecked. Still, only a major miracle could have prevented me from making an occasional error or overlooking an occasional typo. I would appreciate it very much if any such errors in the manual (or, for that matter, in the text) would be reported to me. In addition, for the benefit of other instructors who will receive future editions of this manual, any reasonably written up solutions to presently unsolved problems would also be appreciated.

This manual is posted as a pdf-file with hyperreferences enabled, including a Table of Contents and a Bookmark column listing all chapters and problems. This enables the reader to immediately jump back and forth within the document to locally referenced items (chapters, problems, figures, equations). References to equations, figures, etc., in the book itself can, of course, not be reached in this way. Users of *Adobe Acrobat*® can easily assemble individual problem set solutions from this manual. To make this also possible for instructors with access to only the shareware postscript/pdf viewer *Ghostview*©, each individual problem is started at the top of a new page.

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CHAPTER 1

1.1 Solar energy impinging on the outer layer of earth's atmosphere (usually called "solar constant") has been measured as 1367 W/m^2. What is the solar constant on Mars? (Distance earth to sun = 1.496×10^{11} m, Mars to sun = 2.28×10^{11} m).

Solution

The total energy emitted from the sun $Q = 4\pi R_S^2 \sigma T_{\rm sun}^4$ goes equally into all directions, so that

$$q_{\rm SC}(R) = Q/4\pi R^2 = \sigma T_{\rm sun}^4 \left(\frac{R_S}{R}\right)^2$$

$$\frac{q_{\rm SC\,mars}}{q_{\rm SC\,earth}} = \left(\frac{R_S}{R_M}\right)^2 \bigg/ \left(\frac{R_S}{R_E}\right)^2 = \left(\frac{R_E}{R_M}\right)^2 = \left(\frac{1.8\times10^{11}}{2.28\times10^{11}}\right)^2 = 0.4305$$

$$\boxed{q_{\rm SC\,mars} = 0.4305 \times 1367 = 588\,\mathrm{W/m^2}}$$

1.2 Assuming earth to be a blackbody, what would be its average temperature if there was no internal heating from the core of earth?

Solution
Without internal heating an energy balance for earth gives

$$\begin{aligned} Q_{\text{absorbed}} &= Q_{\text{emitted}} \\ Q_{\text{abs}} &= q_{\text{sol}} \times A_{\text{proj}} = q_{\text{sol}} \pi R_E^2 \\ Q_{\text{em}} &= \sigma T_E^4 A = 4\pi R_E^2 \sigma T_E^4 \end{aligned}$$

Thus

$$T_E = \left(\frac{q_{\text{sol}}}{4\sigma}\right)^{1/4} = \left(\frac{1367\,\text{W/m}^2}{4\times 5.670\times 10^{-8}\,\text{W/m}^2\text{K}^4}\right)^{1/4}$$

$$\boxed{T_E = 279\,\text{K} = 6°\text{C}}$$

1.3 Assuming earth to be a black sphere with a surface temperature of 300 K, what must earth's internal heat generation be in order to maintain that temperature (neglect radiation from the stars, but not the sun) (radius of the earth $R_E = 6.37 \times 10^6$ m).

Solution
Performing an energy balance on earth:

$$\begin{aligned}\dot{Q} &= Q_{\text{emitted}} - Q_{\text{absorbed}} = \sigma T_E^4 A - q_{\text{sol}} A_{\text{proj}} \\ &= 4\pi R_E^2 \sigma T_E^4 - q_{\text{sol}} \pi R_E^2 = \pi R_E^2 \left(4\sigma T_E^4 - q_{\text{sol}}\right) \\ &= \pi \times \left(6.37\times10^6 \text{m}\right)^2 \left[4\times5.670\times10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4} \times 300^4\,\text{K}^4 - 1367 \frac{\text{W}}{\text{m}^2}\right]\end{aligned}$$

$$\boxed{\dot{Q} = 6.00 \times 10^{16}\,\text{W}}$$

1.4 To estimate the diameter of the sun, one may use solar radiation data. The solar energy impinging onto the earth's atmosphere (called the "solar constant") has been measured as 1367 W/m^2. Assuming that the sun may be approximated to have a black surface with an effective temperature of 5777 K, estimate the diameter of the sun (distance sun to earth $S_{ES} \simeq 1.496 \times 10^{11}$ m).

Solution

The sun's energy travels equally into all directions. Thus, on a spherical "shell" around the sun (such that earth's orbit lies on that shell): total energy leaving sun

$$Q_{\text{sun}} = q_{\text{sol}} \times 4\pi\, S_{ES}^2 = \sigma T_{\text{sun}}^4\, 4\pi R_S^2.$$

Thus

$$\begin{aligned} R_S &= S_{ES}\left(\frac{q_{\text{sol}}}{\sigma T_{\text{sun}}^4}\right)^{1/2} \\ &= 1.496 \times 10^{11}\,\text{m}\left(\frac{1367\,\text{W/m}^2}{5.670 \times 10^{-8} \times 5777^4\,\text{W/m}^2}\right)^{1/2} \end{aligned}$$

$$\boxed{R_S = 6.96 \times 10^8\,\text{m}}$$

1.5 Solar energy impinging on the outer layer of earth's atmosphere (usually called "solar constant") has been measured as 1367 W/m^2. Assuming the sun may be approximated as having a surface that behaves like a blackbody, estimate its effective surface temperature. (Distance sun to earth $S_{ES} \simeq 1.496 \times 10^{11}$ m, radius of sun $R_S \simeq 6.96 \times 10^8$ m).

Solution

The sun's energy travels equally into all directions. Thus, on a spherical "shell" around the sun (such that the earth's orbit lies on that shell): total energy leaving sun

$$Q_{\text{sun}} = q_{\text{sol}} \times 4\pi\, S_{ES}^2 = \sigma T_{\text{sun}}^2\, 4\pi R_S^2.$$

Thus

$$\begin{aligned} T_{\text{sun}} &= \left(\frac{q_{\text{sol}}}{\sigma}\right)^{1/4} \left(\frac{S_{ES}}{R_S}\right)^{1/2} \\ &= \left(\frac{1367\,\text{W/m}^2}{5.670\times10^{-8}\,\text{W/m}^2\text{K}^4}\right)^{1/4} \left(\frac{1.496\times 10^{11}}{6.96\times10^8}\right)^{1/2} \end{aligned}$$

$$\boxed{T_{\text{sun}} = 5777\,\text{K}}$$

1.6 A rocket in space may be approximated as a black cylinder of length $L = 20\,\text{m}$ and diameter $D = 2\,\text{m}$. It flies past the sun at a distance of 140 million km such that the cylinder axis is perpendicular to the sun's rays. Assuming that *(i)* the sun is a blackbody at 5777 K and *(ii)* the cylinder has a high conductivity (i.e., is essentially isothermal), what is the temperature of the rocket? (Radius of sun $R_S = 696{,}000\,\text{km}$; neglect radiation from earth and the stars).

Solution

The total energy emitted from the sun $Q = 4\pi R_S^2\,\sigma T_{\text{sun}}^4$ goes equally into all directions, so that the flux arriving at the rocket a distance S_{RS} away from the sun is

$$\begin{aligned} q_R &= Q/4\pi S_{RS}^2 = \sigma T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}}\right)^2 = \text{flux/unit projected area of rocket} \\ \text{Energy emitted} &= A_R \sigma T_R^4 = \left(2\pi R^2 + 2\pi RL\right)\sigma T_R^4 = 2\pi R(R+L)\,\sigma T_R^4 \\ &= \text{Energy absorbed} = A_p q_R = 2RL\,\sigma T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}}\right)^2 \end{aligned}$$

$$T_R^4 = \frac{2RL}{2\pi R(R+L)} T_{\text{sun}}^4 \left(\frac{R_S}{S_{RS}}\right)^2 = \frac{1}{\pi}\frac{20}{1+20}\,(5777\,\text{K})^4 \left(\frac{6.96\times 10^5}{140\times 10^6}\right)^2$$

$$\boxed{T_R = 302\,\text{K.}}$$

1.7 A black sphere of very high conductivity (i.e., isothermal) is orbiting earth. What is its temperature? (Consider the sun but neglect radiation from the earth and the stars). What would be the temperature of the sphere if it were coated with a material that behaves like a black body for wavelengths between 0.4 μm and 3 μm, but does not absorb and emit at other wavelengths?

Solution

Making an energy balance on the sphere

(a)

$$\begin{aligned} Q_{\text{absorbed}} &= Q_{\text{emitted}} \\ Q_{\text{absorbed}} &= q_{\text{sol}} A_{\text{proj}} = q_{\text{sol}} \pi R^2 \\ Q_{\text{emitted}} &= \sigma T^4 A = \sigma T^4 \, 4\pi R^2 \end{aligned}$$

Thus

$$T = \left(\frac{q_{\text{sol}}}{4\sigma}\right)^{1/4} = \left(\frac{1367\,\text{W/m}^2}{4\times 5.670\times 10^{-8}\,\text{W/m}^2\text{K}^4}\right)^{1/4}$$

$$\boxed{T = 279\,\text{K} = 6^\circ\text{C}}$$

(b) If absorption and emission takes place only over the wavelength interval $0.4\,\mu\text{m} < \lambda < 3\,\mu\text{m}$, both Q_{abs} and Q_{em} will be reduced. Since the sphere is spectrally black, we have

$$Q_{\text{em}} = 4\pi R^2 \int_{0.4\,\mu\text{m}}^{3\,\mu\text{m}} E_{b\lambda}\, d\lambda = 4\pi R^2\, \sigma T^4 \left[f(3\,\mu\text{m}\times T) - f(0.4\,\mu\text{m}\times T)\right]$$

Solar emission is black, i.e., $q_{\text{sol}} \propto E_{b\lambda}(T_{\text{sun}})$, or

$$\frac{q_{\text{sol},\lambda}}{q_{\text{sol}}} = \frac{E_{b\lambda}(T_{\text{sun}})}{\sigma T_{\text{sun}}^4},$$

and

$$\begin{aligned} Q_{\text{abs}} &= \pi R^2 \int_{0.4\,\mu\text{m}}^{3\,\mu\text{m}} q_{\text{sol},\lambda}\, d\lambda = \pi R^2 \frac{q_{\text{sol}}}{\sigma T_{\text{sun}}^4} \int_{0.4\,\mu\text{m}}^{3\,\mu\text{m}} E_{b\lambda}(T_{\text{sun}})\, d\lambda \\ &= \pi R^2 q_{\text{sol}} \left[f(3\,\mu\text{m}\times T_{\text{sun}}) - f(0.4\,\mu\text{m}\times T_{\text{sun}})\right] \end{aligned}$$

Thus,

$$\begin{aligned} 4\sigma T^4 \left[f(3\,\mu\text{m}\, T) - f(0.4\,\mu\text{m}\, T)\right] &= q_{\text{sol}} \left[f(3\,\mu\text{m}\, T_{\text{sun}}) - f(0.4\,\mu\text{m}\, T_{\text{sun}})\right] \\ &= 1367\,\text{W/m}^2 \left[f(17,331) - f(2311)\right] \end{aligned}$$

$$\begin{aligned} T^4 \left[f(3\,\mu\text{m}T) - f(0.4\,\mu\text{m}T)\right] &= \frac{1367}{4\times 5.670\times 10^{-8}} (0.9788 - 0.1222) \\ &= 5.1630 \times 10^9\,\text{K}^4 \end{aligned}$$

This nonlinear relation must be solved by iteration, leading to $T \simeq 600\,\text{K}$ ($< 601\,\text{K}$)

$$600^4 \times [0.03934 - 0] = 5.0985 \times 10^9$$

$$\boxed{T = 600\,\text{K}}$$

1.8 A 100 Watt light bulb may be considered to be an isothermal black sphere at a certain temperature. If the light flux (i.e., visible light, $0.4\,\mu\text{m} < \lambda < 0.7\,\mu\text{m}$) impinging on the floor directly (2.5 m) below the bulb is $42.6\,\text{mW/m}^2$, what is the light bulb's effective temperature? What is its efficiency?

Solution
The total heat rate (100 W) leaving the light bulb will go at equal per-unit-area amounts through any (hypothetical) sphere around the bulb. Take a sphere which has the spot on the floor on its surface (radius $h = 2.5\,\text{m}$)

$$\begin{aligned} q_{\text{floor,visible}} &= q_{\text{fv}} = [f(\lambda_2 T) - f(\lambda_1 T)]\,\frac{Q_{\text{bulb}}}{4\pi h^2} \\ f(\lambda_2 T) - f(\lambda_1 T) &= 4\pi h^2 q_{\text{fv}}/Q_{\text{bulb}} = 4\pi\, 2.5^2\text{m}^2\, 42.6\times 10^{-3}\,\frac{\text{W}}{\text{m}^2}\Big/ 100\,\text{W} \\ &= 0.03346 \end{aligned}$$

By trial and error:

$$\begin{aligned} T = 2500\,\text{K} \rightarrow \lambda_1 T &= 0.4\,\mu\text{m}\times 2500\,\text{K} = 1000\,\mu\text{m K} \\ \lambda_2 T &= 0.7\,\mu\text{m}\times 2500\,\text{K} = 1750\,\mu\text{m K} \end{aligned}$$

$$f(\lambda_2 T) - f(\lambda_1 T) = 0.03369 - 0.00032 = 0.03337$$

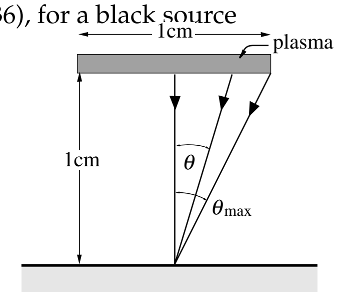
$$\boxed{T = 2500\,\text{K}; \qquad \eta = 3.34\% \qquad \text{is converted into visible light}}$$

The answers for $f(\lambda_1 T)$ and $f(\lambda_2 T)$ are the same whether Appendix C or program planck.exe are employed, since no interpolation is necessary.

1.9 When a metallic surface is irradiated with a highly concentrated laser beam, a plume of plasma (i.e., a gas consisting of ions and free electrons) is formed above the surface that absorbs the laser's energy, often blocking it from reaching the surface. Assume that a plasma of 1 cm diameter is located 1 cm above the surface, and that the plasma behaves like a blackbody at 20,000 K. Based on these assumptions calculate the radiative heat flux and the total radiation pressure on the metal directly under the center of the plasma.

Solution

Radiation emitted from the plasma (≈ black disk) hits the spot below it generating an incoming radiative heat flux and a radiative pressure. From equation (1.36), for a black source



$$\begin{aligned} q_{\text{in}} &= \int_{2\pi} I_b \cos\theta \, d\Omega \\ &= I_b \int_{\Omega\,\text{plasma}} \cos\theta \, d\Omega \\ &= 2\pi\, I_b \int_0^{\theta_{\max}} \cos\theta \, \sin\theta \, d\theta \\ &= E_b \sin^2\theta_{\max} = E_b \times \frac{1}{5} = \frac{5.670\times10^{-8}\times 20,000^4\,\text{W/m}^2}{5} \end{aligned}$$

$$\boxed{q_{\text{in}} = 1.814\times10^9\,\text{W/m}^2 = 181.4\,\text{kW/cm}^2}$$

This should be compared to the maximum flux at the center of an unobstructed laser beam, which may be several MW/cm^2.

The radiation pressure is found similarly, from equation (1.42) as

$$\begin{aligned} p &= \frac{2\pi I_b}{c}\int_0^{\theta_{\max}} \cos^2\theta \,\sin\theta \, d\theta = \frac{2E_b}{3c}\left(1-\cos^2\,\theta_{\max}\right) = \frac{2E_b}{3c}\left[1-\left(\frac{4}{5}\right)^{3/2}\right] \\ &= \frac{2}{3}\,\frac{5.670\times10^{-8}\,20,000^4\,\text{W/m}^2}{3\times10^8\,\text{m/s}}\,[1-0.7155] = 5.735\frac{\text{Ws}}{\text{m}^3} = 5.735\frac{\text{N}}{\text{m}^2} \end{aligned}$$

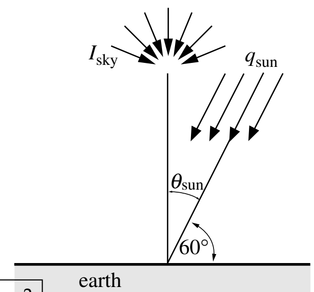
$$\boxed{p = 5.735\,\text{N/m}^2}$$

1.10 Solar energy incident on the surface of the earth may be broken into two parts: A direct component (traveling unimpeded through the atmosphere) and a sky component (reaching the surface after being scattered by the atmosphere). On a clear day the direct solar heat flux has been determined as $q_{\rm sun} = 1000\,{\rm W/m^2}$ (per unit area normal to the rays), while the intensity of the sky component has been found to be diffuse (i.e., the intensity of the sky radiation hitting the surface is the same for all directions) and $I_{\rm sky} = 70\,{\rm W/m^2 sr}$. Determine the total solar irradiation onto earth's surface if the sun is located 60° above the horizon (i.e., 30° from the normal).

Solution

Since earth's surface is tilted away from the sun, less energy per unit surface area hits earth than is carried by the sunshine (per unit area normal to the rays), as seen from Fig. 1-8 (by a factor of $\cos\theta_{\rm sun} = \cos 30°$). Thus from equation (1.36)

$$\begin{aligned} q_{\rm in} &= q_{\rm sun}\cos\theta_{\rm sun} + \int_{2\pi} I_{\rm sky}\cos\theta\, d\Omega \\ &= q_{\rm sun}\cos\theta_{\rm sun} + \int_0^{2\pi}\int_0^{\pi/2} \cos\theta\ \sin\theta\, d\theta\, d\psi \\ &= q_{\rm sun}\cos\theta_{\rm sun} + \pi I_{\rm sky} \\ &= 1000\times\cos 30° + \pi\times 70 = 1086\,{\rm W/m^2} \end{aligned}$$



$$\boxed{q_{\rm in} = 1086\,{\rm W/m^2}}$$

1.11 A window (consisting of a vertical sheet of glass) is exposed to direct sunshine at a strength of $1000\,\mathrm{W/m^2}$. The window is pointing due south, while the sun is in the southwest, 30° above the horizon. Estimate the amount of solar energy that *(i)* penetrates into the building, *(ii)* is absorbed by the window, and *(iii)* is reflected by the window. The window is made of *(a)* plain glass, *(b)* tinted glass, whose radiative properties may be approximated by

$$\rho_\lambda = 0.08 \quad \text{for all wavelengths (both glasses),}$$
$$\tau_\lambda = \begin{cases} 0.90 & \text{for } 0.35\,\mu\text{m} < \lambda < 2.7\,\mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad \text{(plain glass)}$$
$$\tau_\lambda = \begin{cases} 0.90 & \text{for } 0.5\,\mu\text{m} < \lambda < 1.4\,\mu\text{m} \\ 0 & \text{for all other wavelengths} \end{cases} \quad \text{(tinted glass).}$$

(c) By what fraction is the amount of visible light ($0.4\,\mu\mathrm{m} < \lambda < 0.7\,\mu\mathrm{m}$) reduced, if tinted rather that plain glass is used? How would you modify this statement in the light of Fig. 1-11?

Solution

If we want to determine how much sunshine penetrates through the window (for the entire window area, or per unit window area), we need to determine the angle between a normal to the window ($\hat{\mathbf{n}}$) and a vector pointing to the sun ($\hat{\mathbf{s}}$), or $\cos\beta = \hat{\mathbf{n}}\cdot\hat{\mathbf{s}}$. From geometric relations it follows that $\cos\beta = \cos 30°\times\cos 45° = 0.6124$. Thus sunshine on the window, per unit window area, is $q_{\rm in} = 612.4\,\mathrm{W/m^2}$. Of that the fraction 0.08 is reflected, i.e.,

$$q_{\rm ref} = 0.08 \times 612.4 = 49.0\,\mathrm{W/m^2} \text{ of window area.}$$

(a) Plain glass: The transmitted sunshine may be calculated as

$$\begin{aligned} q_{\rm trans} &= \int_0^\infty \tau_\lambda\, q_{{\rm in},\lambda}\, d\lambda = 0.90 \int_{0.35\,\mu\rm m}^{2.7\,\mu\rm m} q_{{\rm in},\lambda}\, d\lambda \\ &= 0.90\, q_{\rm in}\left[f(2.7\,\mu\mathrm{m}\, T_{\rm sun}) - f(0.35\,\mu\mathrm{m}\, T_{\rm sun})\right] \\ &= 0.90 \times 612.4\,\mathrm{W/m^2}\left[f(15{,}598\,\mu\mathrm{m\,K}) - f(2022\,\mu\mathrm{m\,K})\right] \\ &= 0.90 \times 612.4 \times (0.9720 - 0.0702) = 497.0\,\mathrm{W/m^2} \end{aligned}$$

It follows that

$$\boxed{q_{\rm abs} = q_{\rm in} - q_{\rm ref} - q_{\rm trans} = 612.4 - 49.0 - 497.4 = 66.0\,\mathrm{W/m^2}}$$

(b) Tinted glass: Similarly,

$$\begin{aligned} q_{\rm trans} &= 0.90 \times q_{\rm in}\left[f(1.4\,\mu\mathrm{m} \times 5777\,\mathrm{K}) - f(0.5\,\mu\mathrm{m} \times 5777\,\mathrm{K})\right] \\ &= 0.90 \times 612.4 \times \left[f(8088) - f(2889)\right]\ \mathrm{W/m^2} \\ &= 0.90 \times 612.4 \times (0.8597 - 0.2481) = 337.1\,\mathrm{W/m^2}, \end{aligned}$$

$$\boxed{q_{\rm trans} = 337.1\,\mathrm{W/m^2}}$$

i.e., heat gain is reduced by $160\,\mathrm{W/m^2}$ (or 32%).

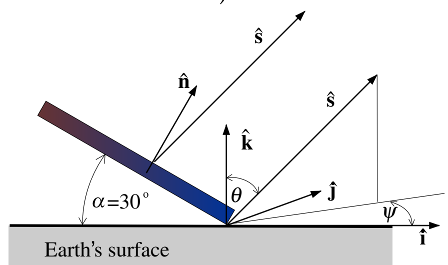
(c) The tinted glass loses some visible light, since it is not transmissive for $0.4\,\mu\mathrm{m} < \lambda < 0.5\,\mu\mathrm{m}$. The fractional reduction is

$$\begin{aligned} \text{fractional reduction} &= \frac{0.90\, q_{\rm in}\left[f(0.5\,\mu\mathrm{m}\, T_{\rm sun}) - f(0.4\,\mu\mathrm{m}\, T_{\rm sun})\right]}{0.90\, q_{\rm in}\left[f(0.7\,\mu\mathrm{m}\, T_{\rm sun}) - f(0.4\,\mu\mathrm{m}\, T_{\rm sun})\right]} \\ &= \frac{f(2889) - f(2311)}{f(4044) - f(2311)} = \frac{0.2481 - 0.1222}{0.4888 - 0.1222} \end{aligned}$$

$$\boxed{\text{fractional reduction} = 34.3\%}$$

Therefore, the tinted glass appears to cause an equivalent loss of visible light. However, Fig. 1-11 shows that the human eye is not very responsive to wavelengths below $\simeq 0.5\,\mu\mathrm{m}$, so that the actual reduction of visible light is considerably less.

1.12 On an overcast day the directional behavior of the intensity of solar radiation reaching the surface of the earth after being scattered by the atmosphere may be approximated as $I_{sky}(\theta) = I_{sky}(\theta = 0)\cos\theta$, where θ is measured from the surface normal. For a day with $I_{sky}(0) = 100\,\text{W/m}^2\text{sr}$ determine the solar irradiation hitting a solar collector, if the collector is *(a)* horizontal, *(b)* tilted from the horizontal by 30°. Neglect radiation from the earth's surface hitting the collector (by emission or reflection).



Solution

(a) For a horizontal collector the solar irradiation is readily determined from equation (1.36) as

$$
\begin{aligned}
q_{in} &= \int_0^{2\pi}\int_0^{\pi/2} (I_0\cos\theta)\cos\theta\sin\theta\,d\theta\,d\psi \\
&= 2\pi I_0 \int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta = \frac{2\pi}{3}I_0 = \frac{2\pi}{3}\times 100
\end{aligned}
$$

$$\boxed{q_{in} = 209.4\,\text{W/m}^2}$$

(b) To evaluate q_{in} for a tilted collector, equation (1.36) becomes

$$q_{in} = \int_{\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}>0} I_0(\hat{\mathbf{k}}\cdot\hat{\mathbf{s}})(\hat{\mathbf{n}}\cdot\hat{\mathbf{s}})\,d\Omega,$$

where $\hat{\mathbf{k}}\cdot\hat{\mathbf{s}} = \cos\theta$ gives the variation of sky intensity with sky polar angle θ, while $\hat{\mathbf{n}}\cdot\hat{\mathbf{s}} = \cos\theta'$ is the cosine of the angle between the surface normal and a unit vector pointing into the sky, $\hat{\mathbf{s}}$. This unit vector may be expressed in terms of $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$, $\hat{\mathbf{k}}$ (unit vectors for an earth-bound coordinate system), with

$$\hat{\mathbf{\imath}}' = \cos\alpha\,\hat{\mathbf{\imath}} - \sin\alpha\,\hat{\mathbf{k}},\ \hat{\mathbf{\jmath}}' = \hat{\mathbf{\jmath}},\ \hat{\mathbf{n}} = \sin\alpha\,\hat{\mathbf{\imath}} + \cos\alpha\,\hat{\mathbf{k}},$$

as

$$
\begin{aligned}
\hat{\mathbf{s}} &= \sin\theta\cos\psi\,\hat{\mathbf{\imath}} + \sin\theta\sin\psi\,\hat{\mathbf{\jmath}} + \cos\theta\,\hat{\mathbf{k}} \\
&= \sin\theta'\cos\psi'\,\hat{\mathbf{\imath}}' + \sin\theta'\sin\psi'\,\hat{\mathbf{\jmath}}' + \cos\theta'\,\hat{\mathbf{n}}.
\end{aligned}
$$

Substituting for $\hat{\mathbf{\imath}}'$, $\hat{\mathbf{\jmath}}'$ and $\hat{\mathbf{n}}$, $\hat{\mathbf{s}}(\theta', \psi')$ becomes

$$
\begin{aligned}
\hat{\mathbf{s}} = (\sin\theta'\cos\psi\cos\alpha\ &+\ \cos\theta'\sin\alpha)\,\hat{\mathbf{\imath}} + \sin\theta'\sin\psi'\,\hat{\mathbf{\jmath}} \\
&+\ (\cos\theta'\cos\alpha - \sin\theta'\cos\psi'\sin\alpha)\,\hat{\mathbf{k}}.
\end{aligned}
$$

Equation (1.36) may be evaluated by evaluating $\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}$ in collector coordinates (i.e., integrating over θ' and ψ'), or by determining $\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}$ in global coordinates (integrating over θ and ψ). Choosing the former, one obtains

$$q_{in} = \int_{\substack{\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}>0\\ \hat{\mathbf{n}}\cdot\hat{\mathbf{s}}>0}} I_0(\cos\theta'\cos\alpha - \sin\theta'\cos\psi'\sin\alpha)\cos\theta'\sin\theta'\,d\theta'\,d\psi'$$

The integration limits imply that only directions above the collector ($0 < \theta < \pi/2$) and above the horizontal

$$\hat{\mathbf{k}}\cdot\hat{\mathbf{s}} = \cos\theta = \cos\theta'\cos\alpha - \sin\theta'\cos\psi'\sin\alpha > 0$$

are to be considered. Solving the last expression for $\cos\psi'$ yields

$$\cos\psi' < \cot\alpha\cot\theta',$$

and using symmetry

$$q_{\rm in} = 2I_0 \int_0^{\pi/2} \int_{\cos^{-1}(\cos\alpha\,\cos\theta')}^{\pi} (\cos\theta'\,\cos\alpha - \sin\theta'\,\cos\psi'\,\sin\alpha) \times \cos\theta'\,\sin\theta'\,d\psi'\,d\theta'.$$

Any radiation from earth's surface hits the collector with a polar angle of $\theta' > 60°$, making a negligible contribution to $q_{\rm in}$. Thus, as a first approximation, the lower limit for ψ' may be replaced by 0, such that

$$\begin{aligned} q_{\rm in} &= 2I_b \int_0^{\pi/2} \int_0^{\pi} (\cos\theta'\,\cos\alpha - \sin\theta'\,\cos\psi'\,\sin\alpha)\,\cos\theta'\,\sin\theta'\,d\psi'\,d\theta' \\ &= \frac{2\pi}{3} I_0\,\cos\alpha = \frac{2\pi}{3} \times 100 \times \cos 30° = 181.4\,{\rm W/m^2}. \end{aligned}$$

$$\boxed{q_{\rm in} = 181.4\,{\rm W/m^2}}$$

This number is a little bit too low because the additionally considered radiation from the earth has a $\hat{\mathbf{k}} \cdot \hat{\mathbf{s}} = \cos\theta < 0$.

1.13 A 100 W light bulb is rated to have a total light output of 1750 lm. Assuming the light bulb to consist of a small, black, radiating body (the light filament) enclosed in a glass envelope (with a transmittance $\tau_g = 0.9$ throughout the visible wavelengths), estimate the filament's temperature. If the filament has an emittance of $\epsilon_f = 0.7$ (constant for all wavelengths and directions), how does it affect its temperature?

Solution

The total heat rate leaving the light bulb by radiation (going equally into all directions) is

$$Q_{\text{bulb}} = \tau\, Q_{\text{fil}} = \tau(A_{\text{fil}}\,\epsilon_{\text{fil}}\,\sigma T_{\text{fil}}^4).$$

Thus the radiative heat flux on a large, hypothetical sphere of radius R surrounding the bulb is

$$q(R) = \frac{Q_{\text{bulb}}}{A_{\text{sphere}}} = \frac{A_{\text{fil}}}{A_{\text{sphere}}}\tau\epsilon_{\text{fil}}\,\pi I_b(T_{\text{fil}}) = I(R)\,d\Omega_{\text{fil}}$$

where $I(R)$ indicates that the flux at R is coming from a very small solid angle normal to the surface and is, therefore, simply a normal intensity multiplied by the small solid angle.

From equation (1.47) the luminous flux at R is

$$q_{\text{lum}}(R) = L d\Omega_{\text{fil}} = 286\frac{\text{lm}}{\text{W}}\int_{0.4\,\mu\text{m}}^{0.7\,\mu\text{m}} I_\lambda(R)\,d\lambda\,d\Omega_{\text{fil}}$$

With gray (i.e., constant) values for ϵ_{fil} and τ (at least over the visible wavelengths), this makes $I_\lambda(R) \propto I_{b\lambda}(T_{\text{fil}})$, and

$$\begin{aligned}
q_{\text{lum}}(R) &= 286\frac{\text{lm}}{\text{W}}I(R)\,d\Omega_{\text{fil}}\left[f(0.7\,\mu\text{m}\,T_{\text{fil}}) - f(0.4\,\mu\text{m}\,T_{\text{fil}})\right].\\
Q_{\text{lum}} &= q_{\text{lum}}(R)A_{\text{sphere}}\\
&= 286\frac{\text{lm}}{\text{W}}A_{\text{fil}}\tau\epsilon_{\text{fil}}\,\pi I_b(T_{\text{fil}})\left[f(0.7\,\mu\text{m}\,T_{\text{fil}}) - f(0.4\,\mu\text{m}\,T_{\text{fil}})\right]\\
&= 286\frac{\text{lm}}{\text{W}}\pi\tau\,Q_{\text{fil}}\,\left[f(0.7\,\mu\text{m}\,T_{\text{fil}}) - f(0.4\,\mu\text{m}\,T_{\text{fil}})\right],
\end{aligned}$$

where Q_{lum} is the total luminous flux leaving the bulb. Thus

$$\begin{aligned}
f(0.7\,T_{\text{fil}}) - f(0.4\,T_{\text{fil}}) &= \frac{Q_{\text{lum}}}{286(\text{lm/W})\,\pi\tau\,Q_{\text{fil}}} = \frac{1750\,\text{lm}}{286(\text{lm/W})\,\pi\times 0.9\times 100\,\text{W}}\\
&= 0.02164.
\end{aligned}$$

By trial and error, one finds from Appendix C, for $T_{\text{fil}} \simeq 2320\,\text{K}$

$$f(0.7\times 2320) - f(0.4\times 2320) = 0.02165 - 0.00013 = 0.02152.$$

$$\boxed{T_{\text{fil}} \simeq 2320\,\text{K}}$$

This expression does not include the filament emittance: $\epsilon_f = 0.7$ would leave T_{fil} unchanged (however, A_{fil} would have to increase by a factor of $1/0.7 = 1.43$, in order to emit 100 W of power).

1.14 A *pyrometer* is a device with which the temperature of a surface may be determined remotely by measuring the radiative energy falling onto a detector. Consider a black detector of 1 mm × 1 mm area, that is exposed to a 1 cm^2 hole in a furnace located a distance of 1 m away. The inside of the furnace is at 1500 K and the intensity escaping from the hole is essentially blackbody intensity at that temperature. *(a)* What is the radiative heat rate hitting the detector? *(b)* Assuming that the pyrometer has been calibrated for the situation in *(a)*, what temperature would the pyrometer indicate if the nonabsorbing gas between furnace and detector were replaced by one with an (average) absorption coefficient of $\kappa = 0.1\,\text{m}^{-1}$?

Solution

(a) A total radiative intensity of $I_b(T_{\text{furnace}})$ leaves the hole, equally into all directions. From the definition of intensity the total heat rate hitting the detector—assuming the detector to be directly opposite the hole—is

$$Q_d = A_{\text{hole}} I_b(T_f)\, d\Omega_{\text{hole-detector}} = A_{\text{hole}} I_b(T_f) \frac{A_{\text{detector}}}{S_{\text{hd}}^2}$$

$$Q_d = \frac{A_{\text{hole}}\, \sigma T_f^4\, A_{\text{detector}}}{\pi\, S_{\text{hd}}^2}$$

$$= \frac{1\,\text{cm}^2 \times 5.670 \times 10^{-12}\,\text{W/cm}^2\,\text{K}^4 \times 1500^4\,\text{K}^4 \times 1\,\text{mm}^2}{\pi\, 1\,\text{m}^2}$$

$$\boxed{Q_d = 9.137 \times 10^{-6}\,\text{W} = 9.137\,\mu\text{W}}$$

(b) The signal would now be attenuated by a factor of $\exp(-\kappa S_{\text{hd}})$, or

$$Q_d^* = Q_d \exp(-\kappa S_{\text{hd}}) = 9.137\,\mu\text{W} \times \exp(-0.1\,\text{m}^{-1} \times 1\,\text{m})$$
$$= 8.2675\,\mu\text{W}$$

Since the d calibration is $Q_d \propto T_f^4$ we have

$$\frac{Q_d}{T_f^4} = \frac{Q_d^*}{T_f^{*4}} \rightarrow T_f^{*4} = \frac{Q_d^*}{Q_d} T_f^4$$

$$\boxed{T_f^* = \left(\frac{8.2675}{9.137}\right)^{1/4} 1500 = 1463\,\text{K},}$$

i.e., the measurement would be wrong by 37 K.

1.15 Consider a pyrometer, which also has a detector area of 1mm × 1mm, which is black in the wavelength range $1.0\,\mu\text{m} \le \lambda \le 1.2\,\mu\text{m}$, and perfectly reflecting elsewhere. In front of the detector is a focussing lens ($f = 10\,\text{cm}$) of diameter $D = 2\,\text{cm}$, and transmissivity of $\tau_l = 0.9$ (around 1μm). In order to measure the temperature inside a furnace, the pyrometer is focussed onto a hot black surface inside the furnace, a distance of 1 m away from the lens.

(a) How large a spot on the furnace wall does the detector see? (Remember that geometric optics dictates

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}; \qquad M = \frac{h\,(\text{detector size})}{H\,(\text{spot size})} = \frac{v}{u},$$

where $u = 1\text{m}$ is the distance from lens to furnace wall, and v is the distance from lens to detector.)

(b) If the temperature of the furnace wall is 1200 K, how much energy is absorbed by the detector per unit time?

(c) It turns out the furnace wall is not really black, but has an emittance of $\epsilon = 0.7$ (around 1μm). Assuming there is no radiation reflected from the furnace surface reaching the detector, what is the true surface temperature for the pyrometer reading of case *(b)*?

(d) To measure higher temperatures pyrometers are outfitted with filters. If a $\tau_f = 0.7$ filter is placed in front of the lens, what furnace temperature would provide the same pyrometer reading as case *(b)*?

Solution

(a) From geometric optics

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{1}{10\text{cm}} - \frac{1}{100\text{cm}} = \frac{9}{100\text{cm}}; \; v = 11.1\text{cm}$$

$$H = h\frac{u}{v} = 1\text{mm} \times \frac{100}{11.1} = 9\text{mm}.$$

Thus, the spot seen by the pyrometer is 9mm × 9mm in size.

(b) Energy leaving spot, intercepted by detector (or, by conservation of energy, energy intercepted by detector, coming from spot) is

$$Q_d = I_{b12}(T)A_H\Omega_{Hl}\tau_l = I_{d12}A_h\Omega_{hl},$$

where $I_{b12}(T) = \int_{\lambda_1}^{\lambda_2} I_{b\lambda}(T)d\lambda$ is emitted intensity from the spot on the furnace wall, I_{d12} is intensity absorbed by the detector, and Ω_{Hl} and Ω_{hl} are the solid angles with which the lens is seen from H and h, respectively. Since H and h are small compared to f and $D = 2R$, we simply evaluate the solid angles from the center of spot H and detector h:

$$\Omega_{Hl} \simeq \frac{\pi R^2}{u^2}; \; \Omega_{hl} = \frac{\pi R^2}{v^2},$$

and

$$A_H\Omega_{hl} = \pi R^2\left(\frac{H}{u}\right)^2 = \pi R^2\left(\frac{h}{v}\right)^2 = A_h\Omega_{hl} = \pi 1\text{cm}^2\left(\frac{.9}{100}\right)^2 = 2.545 \times 10^{-4}\text{cm}^2$$

Therefore, $I_{b12}(T)\tau_l = I_{d12}$, i.e., intensity hitting detector is the *same* as emitted from furnace, except for the attenuation through the lens: total energy is concentrated on a smaller spot by increasing the solid angle. Finally,

$$\begin{aligned} I_{b12} &= \frac{\sigma T^4}{\pi}[f(\lambda_2 T) - f(\lambda_1 T)] = \frac{5.67\times 10^{-8} \times 1200^4}{\pi}\frac{W}{\text{m}^2}[\underbrace{f(1440)}_{0.00961} - \underbrace{f(1200)}_{0.00213}] \\ &= 280\frac{W}{\text{m}^2} \\ &\simeq I_{b\lambda}(1.1\mu\text{m}, 1200\text{k})\Delta\lambda = (E_{b\lambda}/T^5)(T^5/\pi)\Delta\lambda = 1.7228 \times 10^{-12} \times \frac{1200^5}{\pi}0.2\frac{W}{\text{m}^2} \\ &= 273\frac{W}{\text{m}^2} \end{aligned}$$

and

$$\boxed{Q_d = 280\frac{W}{\text{m}^2}2.545\times10^{-8}\text{m}^2\times0.9 = 6.41\mu\text{W}}$$

(c) The energy hitting detector remains the same and, therefore, so does the intensity emitted from the spot:

$$\epsilon I_{b12}(T_a)(\text{actual}) = I_{b12}(T_p = 1200\text{K})(\text{perceived})$$

or, if we assume the blackbody intensity over the detector range can be approximated by the value at 1.1μm,

$$\frac{\epsilon}{e^{C_2/\lambda T_a}-1} \simeq \frac{1}{e^{C_2/\lambda T_p}-1},$$

leading to

$$\begin{aligned} T_a &= \frac{C_2}{\lambda}\Big/\ln\{1+\epsilon[e^{C_2/\lambda T_p}-1]\} \\ &= \frac{14,388\mu\text{mK}}{1.1\mu\text{m}}\Big/\ln\{1+0.7[e^{14{,}388/1.1\times1200}-1]\} \end{aligned}$$

or

$$\boxed{T_a = 1241\text{K}}.$$

At these wavelengths Wien's law holds almost exactly, i.e., we may drop the two "1"s from the above equation, and

$$T_a \simeq \frac{C_2}{\lambda}\Big/\ln(\epsilon e^{C_2/\lambda T_p}) = \frac{C_2/\lambda}{\frac{C_2}{\lambda T_p}+\ln\epsilon} = \frac{T_p}{1+\frac{\lambda T_p}{C_2}\ln\epsilon}$$

which again leads to 1241k.

(d) Similar to *(c)* we get $\tau_f I_{b12}(T_a) = I_{b12}(T_p)$ and, since $\tau_f = \epsilon_{(c)}$, the answer is the same, i.e,

$$\boxed{T_a = 1241\text{K}}.$$

Thus, a much stronger filter is needed to really extend the pyrometer's range.

CHAPTER 2

2.1 Show that for an electromagnetic wave traveling through a dielectric ($m_1 = n_1$), impinging on the interface with another, optically less dense dielectric ($n_2 < n_1$), light of any polarization is totally reflected for incidence angles larger than $\theta_c = \sin^{-1}(n_2/n_1)$.
Hint: Use equations (2.105) with $k_2 = 0$.

Solution
Equations (2.105) become for $k_2 = 0$,

$$\eta_0\, n_1 \sin\theta_1 = w_t' \sin\theta_2,$$
$$w_t'^2 - w_t''^2 = \eta_0^2\, n_2^2,$$
$$w_t'\, w_t'' \cos\theta_2 = 0.$$

The last of these relations dictates that either $w_t'' = 0$ or $\cos\theta_2 = 0$ ($w_t' = 0$ is not possible since—from the first relation—this would imply $\eta_0\, n_1 \sin\theta_1 = 0$ which is known not to be true).

$w_t'' = 0$**:** Substituting this into the second relation leads to $w_t' = \eta_0\, n_2$, and the first leads to $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Since $n_1 > n_2$ this is a legitimate solution only for $\sin\theta_1 \leq (n_2/n_1)$, or $\theta_1 \leq \theta_c = \sin^{-1}(n_2/n_1)$.

$\cos\theta_2 = 0$ ($\theta_2 = \pi/2$)**:** Substituting the first relation into the second gives

$$w_t'' = \eta_0 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2},$$

i.e., a legitimate nonzero solution for $n_1^2 \sin^2\theta_1 - n_2^2 \geq 0$ or $\theta_1 \geq \theta_c$. Inspection of the reflection coefficients, equations (2.109), shows that

$$\widetilde{r}_\parallel = \frac{i n_1^2\, w_t'' + n_2^2\, w_i' \cos\theta_1}{i n_1^2\, w_t'' - n_2^2\, w_i' \cos\theta_1}, \quad \widetilde{r}_\perp = \frac{w_i' \cos\theta_1 + i w_t''}{w_i' \cos\theta_1 - i w_t''}$$

Since, in both reflection coefficients, there are no sign changes within the real and imaginary parts, it follows readily that

$$\rho_\parallel = \widetilde{r}_\parallel \widetilde{r}_\parallel^* = \rho_\perp = \widetilde{r}_\perp \widetilde{r}_\perp^* = 1.$$

2.2 Derive equations (2.109) using the same approach as in the development of equations (2.89) through (2.92). Hint: Remember that within the absorbing medium, $\mathbf{w} = \mathbf{w}' - i\mathbf{w}'' = w'\hat{\mathbf{s}} - iw''\hat{\mathbf{n}}$; this implies that $\mathbf{E}_0$ is *not* a vector normal to $\hat{\mathbf{s}}$. It is best to assume $\mathbf{E}_0 = E_\parallel \hat{\mathbf{e}}_\parallel + E_\perp \hat{\mathbf{e}}_\perp + E_s \hat{\mathbf{s}}$.

Solution

Inside the absorbing medium $\mathbf{w}_t = \mathbf{w}'_t - i\mathbf{w}''_t = w'_t\hat{\mathbf{s}}_t - iw''_t\hat{\mathbf{n}}$, and the electric field vector does not lie in a plane normal to $\hat{\mathbf{s}}$. Thus, we assume a general three-dimensional representation, or

$$\mathbf{E}_0 = E_\parallel \hat{\mathbf{e}}_\parallel + E_\perp \hat{\mathbf{e}}_\perp + E_s \hat{\mathbf{s}}.$$

Following the development for nonabsorbing media, equations (2.77) through (2.88), then leads to

$$\nu\mu\mathbf{H}_0 = \mathbf{w}\times\mathbf{E}_0 = (w'\hat{\mathbf{s}} - iw''\hat{\mathbf{n}})\times(E_\parallel \hat{\mathbf{e}}_\parallel + E_\perp \hat{\mathbf{e}}_\perp + E_s \hat{\mathbf{s}}).$$

This formulation is valid for the transmitted wave, but also for the incident wave ($w''_i = 0$) and reflected wave ($w''_r = 0$, $w'_r = -w'_i$). The contribution from E_s vanishes for incident and reflected wave. Using the same vector relations as given for the nonabsorbing media interface, one obtains

$$\nu\mu\mathbf{H}_0 = w'(E_\parallel\hat{\mathbf{e}}_\perp - E_\perp\hat{\mathbf{e}}_\parallel) - iw''\,(E_\parallel\hat{\mathbf{e}}_\perp\cos\theta - E_\perp\hat{\mathbf{t}} + E_s\hat{\mathbf{e}}_\perp\sin\theta).$$

For the interface condition (with $\nu\mu$ the same everywhere)

$$\nu\mu H_0\times\hat{\mathbf{n}} = w'(E_\parallel\hat{\mathbf{t}} + E_\perp\hat{\mathbf{e}}_\perp\cos\theta) - iw''\,(E_\parallel\cos\theta\hat{\mathbf{t}} + E_\perp\hat{\mathbf{e}}_\perp + E_s\sin\theta\hat{\mathbf{t}}).$$

Thus, from equation (2.78)

$$\begin{aligned} &w'_i(E_{i\parallel}\hat{\mathbf{t}} + E_{i\perp}\hat{\mathbf{e}}_\perp\cos\theta_1) - w'_i(E_{r\parallel}\hat{\mathbf{t}} + E_{r\perp}\hat{\mathbf{e}}_\perp\cos\theta_1)\\ &\quad = w'_t(E_{t\parallel}\hat{\mathbf{t}} + E_{t\perp}\hat{\mathbf{e}}_\perp\cos\theta_2) - iw''_t(E_{t\parallel}\cos\theta_2\hat{\mathbf{t}} + E_{t\perp}\hat{\mathbf{e}}_\perp + E_{ts}\sin\theta_2\hat{\mathbf{t}}) \end{aligned}$$

or

$$w'_i(E_{i\parallel} - E_{r\parallel}) = (w'_t - iw''_t\cos\theta_2)\,E_{t\parallel} - iw''_t\sin\theta_2 E_{ts} \tag{2.2-A}$$

$$w'_i(E_{i\perp} - E_{r\perp})\cos\theta_1 = (w'_t\cos\theta_2 - iw''_t)\,E_{t\perp} \tag{2.2-B}$$

Similarly, from equation (2.77),

$$\mathbf{E}_0\times\hat{\mathbf{n}} = (E_\parallel\hat{\mathbf{e}}_\parallel + E_\perp\hat{\mathbf{e}}_\perp + E_s\hat{\mathbf{s}})\times\hat{\mathbf{n}} = -E_\parallel\hat{\mathbf{e}}_\perp\cos\theta + E_\perp\hat{\mathbf{t}} - E_s\hat{\mathbf{e}}_\perp\sin\theta$$

and

$$(E_{i\parallel} + E_{r\parallel})\cos\theta_1 = E_{t\parallel}\cos\theta_2 + E_{ts}\sin\theta_2 \tag{2.2-C}$$

$$E_{i\perp} + E_{r\perp} = E_{t\perp} \tag{2.2-D}$$

These four equations have 5 unknowns ($E_{r\parallel}$, E_{rs}, $E_{t\parallel}$, $E_{t\perp}$, and E_{ts}), and an additional condition is needed, e.g., equation (2.23) or equation (2.64). Choosing equation (2.23) we obtain, inside the absorbing medium,

$$\begin{aligned} \mathbf{w}\cdot\mathbf{E}_0 = 0 &= (w'_t\hat{\mathbf{s}} - iw''_t\hat{\mathbf{n}})\cdot(E_{t\parallel}\hat{\mathbf{e}}_{t\parallel} + E_{t\perp}\hat{\mathbf{e}}_\perp + E_{ts}\hat{\mathbf{s}})\\ &= w'_tE_{ts} + iw''_t(E_{t\parallel}\sin\theta_2 - E_{ts}\cos\theta_2). \end{aligned} \tag{2.2-E}$$

Eliminating $E_{t\perp}$ from equations (2.2-B) and (2.2-D), with $\widetilde{r}_\perp = E_{r\perp}/E_{i\perp}$, gives

$$w'_i(1-\widetilde{r}_\perp)\cos\theta_1 = (w'_t\cos\theta_2 - iw''_t)(1+\widetilde{r}_\perp),$$

or

$$\boxed{\widetilde{r}_\perp = \frac{w'_i\cos\theta_1 - (w'_t\cos\theta_2 - iw''_t)}{w'_i\cos\theta_1 + (w'_t\cos\theta_2 - iw''_t)},}$$

which is identical to equation (2.109).

Now, eliminating E_{ts} from equations (2.2-A) and (2.2-C) [multiplying equation (2.2-C) by iw_t'' and adding]:

$$w_i'(E_{i\|} - E_{r\|}) + iw_t'' \cos\theta_1 (E_{i\|} + E_{r\|}) = w_t' E_{t\|}. \qquad (2.2\text{-F})$$

Eliminating E_{ts} from equations (2.2-C) and (2.2-E) leads to

$$E_{ts} = \frac{-iw_t'' E_{t\|} \sin\theta_2}{w_t' - iw_t'' \cos\theta_2}$$

$$(E_{i\|} + E_{r\|}) \cos\theta_1 = E_{t\|}\left[\cos\theta_2 - \frac{iw_t'' \sin^2\theta_2}{w_t' - iw_t''\cos\theta_2}\right] = E_{t\|}\frac{w_t'\cos\theta_2 - iw_t''}{w_t' - iw_t''\cos\theta_2}.$$

Using this to eliminate $E_{t\|}$ from equation (2.2-F), with $\widetilde{r}_\| = E_{r\|}/E_{i\|}$, gives

$$w_i'(1-\widetilde{r}_\|) + iw_t''\cos\theta_1(1+\widetilde{r}_\|) = w_t'\cos\theta_1(1+\widetilde{r}_\|)\frac{w_t' - iw_t''\cos\theta_2}{w_t'\cos\theta_2 - iw_t''}$$

$$\begin{aligned} w_i'(w_t'\cos\theta_2 - iw_t'')(1-\widetilde{r}_\|) &\\ = [w_t'(w_t' - iw_t''\cos\theta_2) - iw_t''(w_t'\cos\theta_2 - iw_t'')]\cos\theta_1(1+\widetilde{r}_\|) &\\ = \eta_0^2 m_2^2 \cos\theta_1(1+\widetilde{r}_\|), & \end{aligned}$$

$$\boxed{\widetilde{r}_\| = \frac{w_i'(w_t'\cos\theta_2 - iw_t'') - \eta_0^2 m_2^2\cos\theta_1}{w_i'(w_t'\cos\theta_2 - iw_t'') + \eta_0^2 m_2^2\cos\theta_1},}$$

which is the same as equation (2.109).

It is a simple matter to show that other conditions give the same result. For example, from equation (2.64)

$$n_1^2(E_{i\|}\hat{\mathbf{e}}_{i\|}\cdot\hat{\mathbf{n}} + E_{r\|}\hat{\mathbf{e}}_{r\|}\cdot\hat{\mathbf{n}}) = m_2^2(E_{t\|}\hat{\mathbf{e}}_{t\|}\cdot\hat{\mathbf{n}} + E_{ts}\hat{\mathbf{s}}\cdot\hat{\mathbf{n}})$$

or

$$n_1^2(E_{i\|} - E_{r\|})\sin\theta_1 = m_2^2(E_{t\|}\sin\theta_2 - E_{ts}\cos\theta_2), \text{ etc.}$$

2.3 Find the normal spectral reflectivity at the interface between two absorbing media. [Hint: Use an approach similar to the one that led to equations (2.89) and (2.90), keeping in mind that all wave vectors will be complex, but that the wave will be homogeneous in both media, i.e., all components of the wave vectors are colinear with the surface normal].

Solution

Equations (2.19) and (2.20) remain valid for incident, reflected and transmitted waves, with $\mathbf{w} = \mathbf{w}' - i\mathbf{w}'' = (w' - iw'')\,\hat{\mathbf{n}}$ for all three cases. From equation (2.31) $\mathbf{w}\cdot\mathbf{w} = (w' - iw'')^2\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = \eta_0^2 m^2$ it follows that $w' - iw'' = \pm\eta_0\, m$. Thus

$$\begin{aligned} w_i' - iw_i'' &= \eta_0\, m_1, \\ w_r' - iw_r'' &= -\eta_0\, m_1 \quad \text{(reflected wave is moving in a direction of } -\hat{\mathbf{n}}), \\ w_t' - iw_t'' &= \eta_0\, m_2. \end{aligned}$$

From equations (2.23) and (2.24), it follows that the electric and magnetic field vectors are normal to $\hat{\mathbf{n}}$, i.e., tangential to the surface, say $\mathbf{E}_0 = E_0\hat{\mathbf{t}}$. Then, from equation (2.77)

$$(E_i + E_r)\,\hat{\mathbf{t}}\times\hat{\mathbf{n}} = E_t\,\hat{\mathbf{t}}\times\hat{\mathbf{n}},$$

or

$$E_i + E_r = E_t$$

From equation (2.25) $\nu\mu\mathbf{H}_0 = \mathbf{w}\times\mathbf{E}_0 = (w' - iw'')\,E\,\hat{\mathbf{n}}\times\hat{\mathbf{t}}$, and from equation (2.78)

$$n_1(E_i - E_r) = m_2\,E_t.$$

Substituting for E_t and dividing by E_i, with $\widetilde{r} = E_r/E_i$:

$$m_1(1 - \widetilde{r}) = m_2(1 + \widetilde{r})$$

or

$$\widetilde{r} = \frac{m_1 - m_2}{m_1 + m_2}$$

and

$$\rho_n = \widetilde{r}\widetilde{r}^* = \frac{(m_1 - m_2)(m_1 - m_2)^*}{(m_1 + m_2)(m_1 + m_2)^*} = \left|\frac{(n_1 - n_2) + i(k_1 - k_2)}{(n_1 + n_2) + i(k_1 + k_2)}\right|^2$$

$$\boxed{\rho_n = \frac{(n_1 - n_2)^2 + (k_1 - k_2)^2}{(n_1 + n_2)^2 + (k_1 + k_2)^2}}$$

2.4 A circularly polarized wave in air is incident upon a smooth dielectric surface ($n = 1.5$) with a direction of 45° off normal. What are the normalized Stokes' parameters before and after the reflection, and what are the degrees of polarization?

Solution
From the definition of Stokes' parameters the incident wave has degrees of polarization

$$\boxed{\frac{Q_i}{I_i} = \frac{U_i}{I_i} = 0, \quad \frac{V_i}{I_i} = \pm 1,}$$

the sign giving the handedness of the circular polarization. With $E_{r\parallel} = E_{i\parallel} r_\parallel$ and $E_{r\perp} = E_{i\perp} r_\perp$, from equations (2.50) through (2.53):

$$I_r = E_{i\parallel} E_{i\parallel}^* r_\parallel^2 + E_{i\perp} E_{i\perp}^* r_\perp^2 = E_{i\parallel} E_{i\parallel}^* (\rho_\parallel + \rho_\perp) = \frac{1}{2}(\rho_\parallel + \rho_\perp) I_i$$

Since $E_{i\parallel} E_{i\parallel}^* = E_{i\perp} E_{i\perp}^*$ [from equation (2.51)] and $\rho = r^2$.

Similarly,

$$\begin{aligned}
Q_r &= E_{i\parallel} E_{i\parallel}^* r_\parallel^2 - E_{i\perp} E_{i\perp}^* r_\perp^2 = E_{i\parallel} E_{i\parallel}^* (\rho_\parallel - \rho_\perp) \\
U_r &= E_{i\parallel} E_{i\perp}^* r_\parallel r_\perp + E_{i\perp} E_{i\parallel}^* r_\perp r_\parallel = U_i r_\parallel r_\perp = 0 \\
V_r &= i(E_{i\parallel} E_{i\perp}^* - E_{i\perp} E_{i\parallel}^*)\, r_\parallel r_\perp = V_i r_\parallel r_\perp \\
\frac{Q_r}{I_r} &= \frac{\rho_\parallel - \rho_\perp}{\rho_\parallel + \rho_\perp}, \quad \frac{V_r}{I_r} = \frac{2 r_\parallel r_\perp}{\rho_\parallel + \rho_\perp} \frac{V_i}{I_i}.
\end{aligned}$$

From Snell's law

$$\sin\theta_2 = \frac{\sin\theta_1}{n_2}; \ \cos\theta_2 = \sqrt{1 - \frac{\sin^2\theta_1}{n_2^2}} = \sqrt{1 - \frac{0.5}{1.5^2}} = \sqrt{\frac{7}{9}},$$

and from equations (2.89) and (2.90)

$$\begin{aligned}
r_\parallel &= \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} = \frac{\sqrt{7/9} - 1.5\sqrt{1/2}}{\sqrt{7/9} + 1.5\sqrt{1/2}} = -0.0920, \ \rho_\parallel = 0.0085 \\
r_\perp &= \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{\sqrt{1/2} - 1.5\sqrt{7/9}}{\sqrt{1/2} + 1.5\sqrt{7/9}} = -0.3033, \ \rho_\perp = 0.0920
\end{aligned}$$

$$\boxed{\frac{Q_r}{I_r} = \frac{0.0085 - 0.0920}{0.0085 + 0.0920} = -0.8315, \quad \frac{U_r}{I_r} = 0}$$

$$\boxed{\frac{V_r}{I_r} = \pm\frac{2\times 0.0920\times 0.0085}{0.0085 + 0.0920} = \pm 0.5556}$$

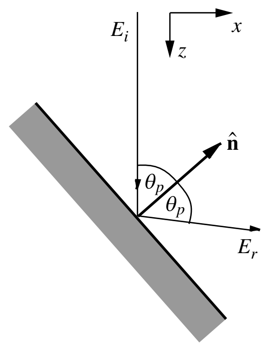
Since the perpendicular polarization is much more strongly reflected, the resulting wave is no longer circularly polarized, but to a large degree linearly polarized (in the perpendicular direction).

2.5 A circularly polarized wave in air traveling along the z-axis is incident upon a dielectric surface ($n = 1.5$). How must the dielectric-air interface be oriented so that the reflected wave is a linearly polarized wave in the y-z-plane?

Solution

From equations (2.50) through (2.53) it follows that $Q_r/I_r = 1$, $U_r = V_r = 0$ (i.e., linear polarization), if either $E_{r\parallel}$ or $E_{r\perp}$ vanish. From Fig. 2-9 it follows that $r_\perp \neq 0$ and, therefore $E_{r\perp} \neq 0$ for all incidence directions, while $r_\parallel = 0$ for $\theta = \theta_p$ (Brewster's angle), or

$$\boxed{\theta_p = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1} 1.5 = 56.31°.}$$



The resulting wave is purely perpendicular-polarized, i.e., $\hat{\mathbf{e}}_\perp$ must lie in the $y-z$ plane, or $\hat{\mathbf{e}}_\parallel$ must be in the $x-z$ plane. Therefore, the surface may be expressed in terms of its surface normal as $\hat{\mathbf{n}} = \hat{\mathbf{i}} \sin\theta_p - \hat{\mathbf{k}} \cos\theta_p = (\hat{\mathbf{i}} - 1.5\hat{\mathbf{k}})\big/\sqrt{3.25}$.

2.6 A polished platinum surface is coated with a 1 μm thick layer of MgO.

(a) Determine the material's reflectivity in the vicinity of $\lambda = 2\,\mu$m (for platinum at $2\,\mu$m $m_{\text{Pt}} = 5.29 - 6.71\,i$, for MgO $m_{\text{MgO}} = 1.65 - 0.0001\,i$).

(b) Estimate the thickness of MgO required to reduce the average reflectivity in the vicinity of $2\,\mu$m to 0.4. What happens to the interference effects for this case?

Solution

(a) The desired overall reflectivity must be calculated from equation (2.124) after determining the relevant reflection coefficient. From equation (2.122)

$$\widetilde{r}_{12} = \frac{1-m_2}{1+m_2} \simeq \frac{1-n_2}{1+n_2} = \frac{1-1.65}{1+1.65} = -0.2453$$

since $k_2 \ll 1$, and $r_{12} = 0.2453$. $\widetilde{r}_{23}$ may also be calculated from equation (2.122) or, more conveniently, from equation (2.126):

$$r_{23}^2 = \frac{(1.65-5.29)^2 + 6.71^2}{(1.65+5.29)^2 + 6.71^2} = 0.6253 \text{ or } r_{23} = 0.7908.$$

Since the real part of $\widetilde{r}_{12} < 0$ it follows that $\delta_{12} = \pi$, while

$$\tan\delta_{23} = \frac{2(1.65\times 6.71 - 5.29\times 10^{-4})}{1.65^2 + 10^{-8} - (5.29^2 + 6.71^2)} = -0.3150.$$

Since the $\Im(r_{23}) > 0$ (numerator) and $\Re(r_{23}) < 0$ (denominator) δ_{23} lies in the second quadrant, $\pi/2 < \delta_{23} < \pi$, or $\delta_{23} = 2.8364$. Also $\zeta_{12} = 4\pi \times 1.65 \times 1\,\mu\text{m}/2\,\mu\text{m} = 10.3673$, and

$$\cos\left[\delta_{12} \pm (\delta_{23} - \zeta_{12})\right] = \cos\left[\pi \pm (2.8364 - 10.3673)\right] = -0.3175.$$

Also $\kappa_2 d = 4\pi \times 10^{-4} \times 1\,\mu\text{m}/2\,\mu\text{m} = 2\pi \times 10^{-4}$ and $\tau = e^{-\kappa_2 d} = 0.9994 \simeq 1$. Thus

$$R = \frac{0.2453^2 + 2\times 0.2453\times 0.7908\times(-0.3175) + 0.7908^2}{1 + 2\times 0.2453\times 0.7908\times(-0.3175) + 0.2453^2\times 0.7908^2}$$

$$\boxed{R = 0.6149.}$$

(b) The cos in the numerator fluctuates between $-1 < \cos < +1$. The average value for R is obtained by dropping the cos-term. Then

$$R_{\text{av}} = \frac{r_{12}^2 + r_{23}^2\tau^2}{1 + r_{12}^2 r_{23}^2 \tau^2},$$

or

$$\tau^2 = \frac{R_{\text{av}} - r_{12}^2}{r_{23}^2(1-r_{12}^2)} = \frac{0.4 - 0.2453^2}{0.7908^2(1-0.2453^2)} = 0.5782,$$

$$d = -\frac{1}{\kappa_2}\ln\tau = -\frac{1}{2\kappa_2}\ln\tau^2 = \frac{-\ln 0.5782}{4\pi\times 10^{-4}\,\mu\text{m}^{-1}} = 43.6\,\mu\text{m}.$$

More accurate is the averaged expression, equation (2.129)

$$R_{\text{av}} = \rho_{12} + \frac{\rho_{23}(1-\rho_{12})^2\tau^2}{1-\rho_{12}\rho_{23}\tau^2}$$

or

$$\tau^2 = \frac{R_{\text{av}} - \rho_{12}}{\rho_{23}\left[(R_{\text{av}}-\rho_{12})\rho_{12} + (1-\rho_{12})^2\right]} = \frac{R_{\text{av}} - \rho_{12}}{\rho_{23}\left[1 - (2-R_{\text{av}})\rho_{12}\right]}$$

$$= \frac{0.4 - 0.2453^2}{0.7908^2\left[1 - 1.6\times 0.2453^2\right]} = 0.6013$$