Chapter 2

Calculus of Variations and Optimal Control

Solution to Problem 2.1

The performance index is

$$\begin{aligned} J &= \int_0^2 [2x^2(t) + \dot{x}^2(t)]dt, \\ &= \int_{t_0}^{t_f} V(x(t), \dot{x}(t))dt \end{aligned}$$

where

$$V = 2x^2(t) + \dot{x}^2(t), \qquad t_0 = 0, \qquad t_f = 2.$$

Using the Euler-Lagrange equation (2.3.15)

$$\begin{aligned} \left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* &= 0 \longrightarrow \\ 4x^*(t) - \frac{d}{dt}(2\dot{x}^*(t)) &= 0 \longrightarrow \\ \ddot{x}^*(t) - 2x^*(t) &= 0. \end{aligned}$$

Solving the previous equation

$$x^*(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$$

where, C_1 and C_2 are constants evaluated using the given boundary conditions

$$x(0) = 0 \longrightarrow C_1 + C_2 = 0$$
$$x(2) = 5 \longrightarrow C_1 e^{2\sqrt{2}} + C_2 e^{-2\sqrt{2}} = 5$$

The final optimal solution is

$$\boxed{x^*(t) = \frac{5}{e^{2\sqrt{2}} - e^{-2\sqrt{2}}}\left[e^{\sqrt{2}t} - e^{-\sqrt{2}t}\right]}$$

or

$$\boxed{x^*(t) = \left[\frac{10}{e^{2\sqrt{2}} - e^{-2\sqrt{2}}}\right] sinh(\sqrt{2}t)}$$

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%% Problem 2.1
%% Solution Using Symbolic Toolbox (STB) in
%% The Student Edition of MATLAB. Version 6
%%
x=dsolve('D2x - 2*x=0','x(0)=0,x(2)=5')

10*exp(2*2^(1/2))/(exp(2*2^(1/2))^2-1)*sinh(2^(1/2)*t)

pretty(x)


                          1/2         1/2
                  exp(2 2    ) sinh(2    t)
               10 ------------------------
                              1/2 2
                      exp(2 2    )  - 1

*********************************************************
```

Solution to Problem 2.2

The performance index is

$$J = \int_{-2}^{0} \left[12tx(t) + \dot{x}^2(t)\right] dt$$
$$= \int_{t_0}^{t_f} V(x(t), \dot{x}(t), t) dt$$

where,

$$V = 12tx(t) + \dot{x}^2(t), \qquad t_0 = -2, \qquad t_f = 0.$$

Using the Euler-Lagrange equation (2.3.15)

$$\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0 \longrightarrow$$
$$12t - \frac{d}{dt}\left(2\dot{x}^*(t)\right) = 0 \longrightarrow$$
$$\ddot{x}^*(t) - 6t = 0.$$

Solving the previous equation

$$x^*(t) = t^3 + C_1 t + C_2$$

where, C_1 and C_2 are determined from the given boundary conditions

$$x(-2) = 3 \longrightarrow (-2)^3 + C_1(-2) + C_2 = 3$$
$$x(0) = 0 \longrightarrow C_2 = 0.$$

The optimal solution is

$$\boxed{x^*(t) = t^3 - \frac{11}{2}t}$$

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%% Problem 2.2b
%% Solution Using Symbolic Toolbox (STB) in
%% The Student Edition of MATLAB. Version 6
%%
x=dsolve('D2x - 6*t=0','x(-2)=3,x(0)=0')
```

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t^3-11/2*t

pretty(x)

  3
 t  - 11/2 t

**********************************************************
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Solution to Problem 2.3

The cost function is

$$\begin{aligned} J &= \int_1^2 \left[\frac{\dot{x}^2(t)}{2t^3}\right] dt \\ &= \int_{t_0}^{t_f} V((\dot{x}(t), t)dt \end{aligned}$$

where,

$$V = \frac{\dot{x}^2(t)}{2t^3}, \qquad t_0 = 1, \qquad t_f = 2.$$

Using the Euler-Lagrange equation (2.3.15)

$$\begin{aligned} \left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* &= 0 \longrightarrow \\ 0 - \frac{d}{dt}\left(\frac{2\dot{x}^*(t)}{2t^3}\right) &= 0 \longrightarrow \\ \frac{\dot{x}^*(t)}{t^3} &= C_1 \text{ or} \\ t\ddot{x}^*(t) - 3\dot{x}^*(t) &= 0. \end{aligned}$$

Solving the previous equation

$$x^*(t) = \frac{C_1}{4}t^4 + C_2.$$

Using the given boundary conditions

$$\begin{aligned} x(1) = 1 &\longrightarrow \frac{C_1}{4} + C_2 = 1 \\ x(2) = 10 &\longrightarrow \frac{C_1}{4}2^4 + C_2 = 10. \end{aligned}$$

The final optimal solution is

$$\boxed{x^*(t) = \frac{3}{5}t^4 + \frac{2}{5}}$$

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*********************************************************
%% Problem 2.3
%% Solution Using Symbolic Toolbox (STB) in
%% The Student Edition of MATLAB. Version 6
%%
x=dsolve('t*D2x - 3*Dx=0','x(1)=1,x(2)=10')

2/5+3/5*t^4

pretty(x)

           4
2/5 + 3/5 t

*********************************************************
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Solution to Problem 2.4

The derivation follows the material in Section 2.3. The problem is to find the *optimal* function $x^*(t)$ for which the functional

$$J(x(t)) = \int_{t_0}^{t_f} V(x(t), \dot{x}(t), \ddot{x}(t), t)dt$$

has a relative *optimum.* It is assumed that the integrand V has continuous first and second partial derivatives w.r.t. all its arguments; t_0 and t_f are fixed (or given a priori) and the end points are fixed, i.e.,

$$x(t = t_0) = x_0, \quad x(t = t_f) = x_f, \quad \dot{x}(t = t_0) = \dot{x}_0, \quad \dot{x}(t = t_f) = \dot{x}_f.$$

We know from Theorem 2.1 that the necessary condition for an optimum is that the *variation of a functional vanishes.* Hence, in our attempt to find the optimum of $x(t)$, we first define the increment for J, obtain its variation and finally apply the fundamental theorem of the calculus of variations (Theorem 2.1).

Thus, the various steps involved in finding the optimal solution are first listed and then discussed in detail.

- **Step 1:** *Assumption of an Optimum*
- **Step 2:** *Variations and Increment*
- **Step 3:** *First Variation*
- **Step 4:** *Fundamental Theorem*
- **Step 5:** *Fundamental Lemma*
- **Step 6:** *Euler-Lagrange Equation*

- **Step 1:** *Assumption of an Optimum:* Let us assume that $x^*(t)$ is the optimum attained for the function $x(t)$. Take some admissible function $x_a(t) = x^*(t) + \delta x(t)$ close to $x^*(t)$, where $\delta x(t)$ is the variation of $x^*(t)$ as shown in Figure 2.4. The function $x_a(t)$ should also satisfy the boundary conditions and hence it is necessary that

$$\delta x(t_0) = \delta x(t_f) = 0, \quad \delta \dot{x}(t_0) = \delta \dot{x}(t_f) = 0.$$

- **Step 2:** *Variations and Increment:* Let us first define the increment as

$$\begin{aligned}\Delta J) &\triangleq J(x^*(t)+\delta x(t), \dot{x}^*(t)+\delta\dot{x}(t), \ddot{x}^*(t)+\delta\ddot{x}(t), t) \\ &\quad -J(x^*(t), \dot{x}^*(t), \ddot{x}^*(t), t) \\ &= \int_{t_0}^{t_f} V\left(x^*(t)+\delta x(t), \dot{x}^*(t)+\delta\dot{x}(t), \ddot{x}^*(t)+\delta\ddot{x}(t), t\right) dt \\ &\quad - \int_{t_0}^{t_f} V(x^*(t), \dot{x}^*(t), \ddot{x}^*(t), t) dt.\end{aligned}$$

which by combining the integrals can be written as

$$\begin{aligned}\Delta J = \int_{t_0}^{t_f} &\left[V\left(x^*(t)+\delta x(t), \dot{x}^*(t)+\delta\dot{x}(t), \ddot{x}^*(t)+\delta\ddot{x}(t), t\right)\right. \\ &\left. -V(x^*(t), \dot{x}^*(t), \ddot{x}^*(t), t)\right] dt.\end{aligned}$$

where,

$$\dot{x}(t)=\frac{dx(t)}{dt}, \quad \delta\dot{x}(t)=\frac{d}{dt}\left\{\delta x(t)\right\}, \quad \delta\ddot{x}(t)=\frac{d^2}{dt^2}\left\{\delta x(t)\right\}$$

Expanding V in the increment in a Taylor series, the increment ΔJ becomes

$$\Delta J = \int_{t_0}^{t_f}\left[\frac{\partial V}{\partial x}\delta x(t)+\frac{\partial V}{\partial \dot{x}}\delta\dot{x}(t)+\frac{\partial V}{\partial \ddot{x}}\delta\ddot{x}(t)\right] dt.$$

Here, the partial derivatives are w.r.t. $x(t)$ and $\dot{x}(t)$ at the optimal condition (*) and * is omitted for simplicity.

- **Step 3:** *First Variation:* Now, we obtain the variation by retaining the terms that are *linear* in $\delta x(t)$ and $\delta\dot{x}(t)$ as

$$\delta J = \int_{t_0}^{t_f}\left[\frac{\partial V}{\partial x}\delta x(t)+\frac{\partial V}{\partial \dot{x}}\delta\dot{x}(t)+\frac{\partial V}{\partial \ddot{x}}\delta\ddot{x}(t)\right] dt.$$

To express the relation for the first variation entirely in terms containing $\delta x(t)$ (since $\delta\dot{x}(t)$ is dependent on $\delta x(t)$), we integrate by parts the term involving $\delta\dot{x}(t)$ as (omitting the arguments in