1. The lifetime (measured in years) of a processor is exponentially distributed, with a mean lifetime of 2 years. You are told that a processor failed sometime in the interval [4, 8] years. Given this information, what is the conditional probability that it failed before it was 5 years old?

**Solution:**
Denote the lifetime of the processor by $T$. Since $E(T) = 2$, $\lambda = 0.5$ and the distribution function of $T$ is $F(t) = 1 - e^{-0.5t}$. Using the conditional probability formula:

$$\Pr(T < 5 \mid 4 \leq T \leq 8) = \frac{\Pr([T < 5] \cap [4 \leq T \leq 8])}{\Pr(4 \leq T \leq 8)}$$

$$= \frac{\Pr(4 \leq T < 5)}{\Pr(4 \leq T \leq 8)}$$

$$= \frac{F(5) - F(4)}{F(8) - F(4)}$$

$$= \frac{e^{-2} - e^{-2.5}}{e^{-2} - e^{-4}} = 0.455$$

2. The lifetime of a processor (measured in years) follows the Weibull distribution, with parameters $\lambda = 0.5$ and $\beta = 0.6$.

(a) What is the probability that it will fail in its first year of operation?

(b) Suppose it is still functional after $t = 6$ years of operation. What is the conditional probability that it will fail in the next year?

(c) Repeat parts (a) and (b) for $\beta = 2$. 
(d) Repeat parts (a) and (b) for $\beta = 1$.

**Solution:**

(a) Denote the lifetime of the processor by $T$. The distribution function of $T$ is $F(t) = 1 - e^{-0.5e^{0.6}}$. The probability that $T$ is no greater than one year is $F(1) = 0.393$.

(b) We use the conditional probability formula:

$$\text{Prob}(6 < T \leq 7 \mid T > 6) = \frac{\text{Prob}(6 < T \leq 7)}{\text{Prob}(T > 6)} = \frac{F(7) - F(6)}{1 - F(6)} = 0.045$$

3. To get a feel for the failure rates associated with the Weibull distribution, plot them for the following parameter values as a function of the time, $t$:

(a) Fix $\lambda = 1$ and plot the failure rate curves for $\beta = 0.5, 1.0, 1.5$.

(b) Fix $\beta = 1.5$ and plot the failure rate curves for $\lambda = 1, 2, 5$.

**Solution:**

The failure rate for the Weibull distribution is

$$\lambda(t) = \lambda \beta t^{\beta-1}$$

![Figure 2.1: The failure rate for the Weibull distribution.](https://www.book4me.xyz/solution-manual-fault-tolerant-systems-koren-krishna/)

4. Write the expression for the reliability $R_{\text{system}}(t)$ of the series/parallel system shown in Figure 2.2, assuming that each of the five modules has a reliability of $R(t)$.

**Solution:**

The system can be decomposed into a series system consisting of one unit with the leftmost
4 blocks and the second unit with the rightmost block. If the reliability of the leftmost 4 blocks is \( R_A(t) \), the system reliability is \( R_A(t)R(t) \).

Now, we calculate \( R_A(t) \). This subsystem consists of a parallel arrangement of one unit consisting of the bottom block and another consisting of the other 3 blocks. If \( R_B(t) \) is the reliability of the top 3 blocks, \( R_A(t) = 1 - (1 - R_B(t))(1 - R(t)) \).

Next, we calculate \( R_B(t) \): this subsystem consists of a series arrangement of one block with another consisting of two blocks in parallel. Hence, we have

\[
R_B(t) = R(t)(1 - (1 - R(t))^2)
\]

Substituting all intermediate results yields

\[
R_{system} = R^5(t) - 3R^4(t) + 2R^3(t) + R^2(t)
\]

5. The lifetime of each of the seven blocks in Figure 2.3 is exponentially distributed with parameter \( \lambda \). Derive an expression for the reliability function of the system, \( R_{system}(t) \), and plot it over the range \( t = [0, 100] \) for \( \lambda = 0.02 \).

\[
\left[ 1 - (1 - R(t))^4 \right] \left[ 1 - (1 - R(t))(1 - R^2(t)) \right] = R^7(t) - 5R^6(t) + 9R^5(t) - 6R^4(t) - 2R^3(t) + 4R^2(t)
\]
where $R(t) = e^{-\lambda t}$.

6. Consider a triplex that produces a one-bit output. Failures that cause the output of a processor to be permanently stuck at 0 or stuck at 1 occur at constant rates $\lambda_0$ and $\lambda_1$, respectively. The voter never fails. At time $t$, you carry out a calculation whose correct output is 0. What is the probability that the triplex will produce an incorrect result? (Assume that stuck-at faults are the only ones that a processor can suffer from, and that these are permanent faults; once a processor has its output stuck at some logic value, it remains stuck at that value forever).

Solution:
The result will be incorrect if at least two of the three nodes produce a 1. The probability that a node is stuck by time $t$ at either 0 or 1 is $1 - e^{-(\lambda_0 + \lambda_1)t}$, and the probability that it is stuck at 1 (and not at 0) is

$$q(t) = \frac{\lambda_1}{\lambda_0 + \lambda_1} \left(1 - e^{-(\lambda_0 + \lambda_1)t}\right)$$

and the answer is

$$q^3(t) + 3q^2(t)(1 - q(t))$$

7. Write the expression for the reliability of a 5MR system and calculate its MTTF. Assume that failures occur as a Poisson process with rate $\lambda$ per node, that failures are independent and permanent, and that the voter is failure-free.

Solution:
Denote by $r(t) = e^{-\lambda t}$ the reliability of an individual node. The reliability of the 5MR system is

$$R_{5MR}(t) = \sum_{i=3}^{5} \binom{5}{i} r^i(t)(1 - r(t))^{5-i}$$

$$= 10r^3(t) - 15r^4(t) + 6r^5(t)$$

$$= 10e^{-3\lambda t} - 15e^{-4\lambda t} + 6e^{-5\lambda t}$$

and the MTTF is

$$\int_{t=0}^{\infty} R_{5MR}(t)dt = \frac{1}{\lambda} \left(\frac{10}{3} - \frac{15}{4} + \frac{6}{5}\right) = \frac{47}{60\lambda}$$

8. Consider an NMR system that produces an eight-bit output. $N = 2m + 1$ for some $m$. Each processor fails at a constant rate $\lambda$ and the failures are permanent. A failed processor produces any of the $2^8$ possible outputs with equal probability. A majority voter is used to produce the overall output, and the voter is assumed never to fail. What is the probability that, at time $t$, a majority of the processors produce the same incorrect output after executing some program?

Solution:
The probability that an individual processor is faulty at time $t$ is $q(t) = 1 - e^{-\lambda t}$. 

The probability that \( n_1 \) out of the \( N \) processors are faulty is

\[
\binom{N}{n_1} q^{n_1}(t)(1 - q(t))^{N-n_1}
\]

The probability that exactly \( n_2 \) out of the \( n_1 \) faulty processors produces the output \( \omega \) is

\[
\binom{n_1}{n_2} 2^{-8n_2}(1 - 2^{-8})^{n_1-n_2}
\]

The probability that a majority of the processors produces \( \omega \) is

\[
\sum_{n_1=m+1}^{N} \sum_{n_2=m+1}^{n_1}\left( \binom{N}{n_1} q^{n_1}(t)(1 - q(t))^{N-n_1}\binom{n_1}{n_2} 2^{-8n_2}(1 - 2^{-8})^{n_1-n_2}\right)
\]

The number of incorrect outputs is \( 2^8 - 1 = 255 \). Hence, the answer is

\[
255 \sum_{n_1=m+1}^{N} \sum_{n_2=m+1}^{n_1}\left( \binom{N}{n_1} q^{n_1}(t)(1 - q(t))^{N-n_1}\binom{n_1}{n_2} 2^{-8n_2}(1 - 2^{-8})^{n_1-n_2}\right)
\]

9. Design a majority voter circuit out of two- and three-input logic gates. Assume that you are voting on one-bit inputs.

**Solution:**
Let the inputs be \( a, b, c \). Then, it is easy to check that the logic expression for the voter output is \( \overline{a}bc + a\overline{b}c + abc \), which simplifies to \( ab + bc + ac \). The circuit is shown in Figure 2.4.

![Figure 2.4: Voter circuit diagram.](image)

10. Derive an expression for the reliability of the voter you designed in the previous question. Assume that, for a given time \( t \), the output of each gate is stuck-at-0 or stuck-at-1 with probability \( P_0 \) and \( P_1 \), respectively (and is fault-free with probability \( 1 - P_0 - P_1 \)). What is the probability that the output of your voter circuit is stuck-at-0 (stuck-at-1) given that the 3 inputs to the voter are fault-free and do change between 000 and 111?
Solution:
The voter will be faulty if any of the gates are malfunctioning; since we can implement it using 4 gates, the reliability is \( R = (1 - p_0 - p_1)^4 \). Note that \( 1 - R \) is NOT the probability that the voter output will be wrong every time.

The probability that the output of the voter will be stuck-at-1 is

\[
\text{Prob}(X \text{ is stuck-at-1 } \mid \text{ the inputs are correct}) = \text{Prob}(\text{OR gate stuck } \mid \text{ the inputs are correct}) \cdot \text{Prob}(\text{OR gate is fault-free} \mid \text{ the inputs are correct})
\]

\[
= \text{Prob}(\text{at least 1 of the AND gates is stuck } \mid \text{ the inputs are correct}) = p_1 + (1 - p_0 - p_1)[1 - (1 - p_1)^3].
\]

The probability that the output of the voter will be stuck-at-0 is

\[
\text{Prob}(X \text{ is stuck-at-0 } \mid \text{ the inputs are correct}) = \text{Prob}(\text{OR gate stuck } \mid \text{ the inputs are correct}) \cdot \text{Prob}(\text{OR gate is fault-free} \mid \text{ the inputs are correct})
\]

\[
= p_0 + (1 - p_0 - p_1)p_0^3.
\]

11. Show that the MTTF of a parallel system of \( N \) modules, each of which suffers permanent failures at a rate \( \lambda \), is \( \text{MTTF}_p = \sum_{k=1}^{N} \frac{1}{k!} \).

Solution:
Let the state of the system be the number of modules that are still functional and let \( T_k \) be the time spent in state \( k \). Then, \( E[T_k] = \frac{1}{k!} \). Since \( \text{MTTF} = \sum_{k=1}^{N} E[T_k] \), the result follows immediately.

Another way to prove it is as follows:
The reliability of the parallel system is

\[
R(t) = 1 - (1 - e^{-\lambda t})^N
\]

Denoting \( x = (1 - e^{-\lambda t}) \) the above expression can be rewritten as

\[
R(t) = (1 - x)(1 + \sum_{k=1}^{N-1} x^k)
\]

The MTTF is calculated from \( \int_0^\infty R(t) \, dt \). Since \( dx = \lambda e^{-\lambda t} \, dt \), we obtain

\[
\text{MTTF} = \frac{1}{\lambda} \int_0^\infty \sum_{k=0}^{N-1} x^k \, dx = \frac{1}{\lambda} \left[ \sum_{k=0}^{N-1} \frac{x^{k+1}}{k+1} \right]_0^1 = \frac{1}{\lambda} \sum_{k=0}^{N-1} \frac{1}{k+1} = \frac{1}{\lambda} \sum_{k=1}^{N} \frac{1}{k}
\]

12. Consider a system consisting of 2 subsystems in series. For improved reliability, you can build subsystem \( i \) as a parallel system with \( k_i \) units, for \( i = 1, 2 \). Suppose permanent failures occur at a constant rate \( \lambda \) per unit.

(a) Derive an expression for the reliability of this system.

(b) Obtain an expression for the MTTF of this system with \( k_1 = 2 \) and \( k_2 = 3 \).