

2.1

$$a_i = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$a_i a_i = 6 \text{ (scalar)}$$

$$(a) \quad a_i a_j = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 4 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \text{ (vector)}$$

$$a_i = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$a_i a_i = 5 \text{ (scalar)}$$

$$(b) \quad a_i a_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 2 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \text{ (vector)}$$

$$a_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$a_i a_i = 6 \text{ (scalar)}$$

$$(c) \quad a_i a_j = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 6 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix} 3 \\ 8 \\ 6 \end{bmatrix} \text{ (vector)}$$

2.2

$$A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, six independent components

$$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
, three independent components

$$A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) = \begin{bmatrix} 1 & 3/2 & 1 \\ 3/2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, six independent components

$$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
, three independent components

$$A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

(c) $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) = \begin{bmatrix} 0 & 5/2 & 1 \\ 5/2 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$, six independent components

$$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji}) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
, three independent components

2.3.

$$A_{ij}B_{ij} = -A_{ji}B_{ji} = -A_{ij}B_{ij} \Rightarrow 2A_{ij}B_{ij} = 0 \Rightarrow A_{ij}B_{ij} = 0$$

2.4

$$\begin{aligned} A_{ij}C_{ij} &= A_{ji}C_{ij} = A_{ij}C_{ji} \\ &= A_{ij}(C_{(ij)} + C_{[ij]}) = A_{ij}C_{(ij)} \end{aligned}$$

$$\begin{aligned} B_{ij}C_{ij} &= -B_{ji}C_{ij} = -B_{ij}C_{ji} \\ &= B_{ij}(C_{(ij)} + C_{[ij]}) = B_{ij}C_{[ij]} \end{aligned}$$

$$\det(B_{ij}) = \begin{vmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{vmatrix} = -B_{12}B_{13}B_{23} + B_{13}B_{12}B_{23} = 0$$

2.5

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ik}$$

$$\delta_{ij}a_{kj} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{12} + \delta_{13}a_{13} & \delta_{11}a_{21} + \delta_{12}a_{22} + \delta_{13}a_{23} & \delta_{11}a_{31} + \delta_{12}a_{32} + \delta_{13}a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{ki}$$

2.6

$$\begin{aligned} \det(A_{ij}) &= \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{231}A_{12}A_{23}A_{31} + \varepsilon_{312}A_{13}A_{21}A_{32} \\ &\quad + \varepsilon_{321}A_{13}A_{22}A_{31} + \varepsilon_{132}A_{11}A_{23}A_{32} + \varepsilon_{213}A_{12}A_{21}A_{33} \\ &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} \\ &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31}) \\ &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \text{ (using the usual first row expansion)} \end{aligned}$$

Likewise the same for the form $\det[A_{ij}] = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

2.7

$$\det[A_{ij}] = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}, \quad \det[B_{ij}] = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = B_{11}B_{22} - B_{12}B_{21}$$

$$\begin{aligned} \Rightarrow \det[A_{ij}]\det[B_{ij}] &= (A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21}) \\ &= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22} \end{aligned}$$

$$\begin{aligned} \det[AB] &= \begin{vmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{vmatrix} \\ &= (A_{11}B_{11} + A_{12}B_{21})(A_{21}B_{12} + A_{22}B_{22}) - (A_{11}B_{12} + A_{12}B_{22})(A_{21}B_{11} + A_{22}B_{21}) \\ &= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22} = \det[A_{ij}]\det[B_{ij}] \end{aligned}$$

Orthogonality Properties :

$$\text{Given } A_{ik}A_{jk} = \delta_{ij} \Rightarrow \det(A_{ik}A_{jk}) = \det \delta_{ij} \Rightarrow (\det A)(\det A) = 1 \Rightarrow \det A = \pm 1$$

2.8

$$\text{Property (2.5.4) } \varepsilon_{ijk}\varepsilon_{pqr} \det[A_{mm}] = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix}$$

$$\text{For the case } A_{ij} = \delta_{ij} \Rightarrow \varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \quad (2.5.5)$$

$$\text{From (2.5.5) with } p = i \Rightarrow \varepsilon_{ijk}\varepsilon_{iqr} = \begin{vmatrix} \delta_{ii} & \delta_{iq} & \delta_{ir} \\ \delta_{ji} & \delta_{jq} & \delta_{jr} \\ \delta_{ki} & \delta_{kq} & \delta_{kr} \end{vmatrix} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq} \quad (2.5.6)$$

$$\text{From (2.5.6) with } q = j \Rightarrow \varepsilon_{ijk}\varepsilon_{ijr} = \delta_{jj}\delta_{kr} - \delta_{jr}\delta_{kj} = 3\delta_{kr} - \delta_{kr} = 2\delta_{kr} \quad (2.5.7)$$

$$\text{From (2.5.7) with } r = k \Rightarrow \varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = 6 \quad (2.5.8)$$

2.9

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_1 = 0, \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 \Rightarrow \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\text{From (2.6.4) } \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2 \times \mathbf{e}_1, \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 = -\mathbf{e}_3 \times \mathbf{e}_2, \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_3 \times \mathbf{e}_1$$

$$\Rightarrow \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k$$

2.10

For 90° rotation about x_2 - axis : $Q_{ij} = \begin{bmatrix} \cos 90^\circ & \cos 90^\circ & \cos 180^\circ \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Cases from Exercise 2.1 :

(a) $a'_i = Q_{ij} a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $|\mathbf{a}| = (a_i a_i)^{1/2} = \sqrt{6}$, $|\mathbf{a}'| = (a'_i a'_i)^{1/2} = \sqrt{6}$

$A'_{ij} = Q_{ip} Q_{jq} A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $A_{ii} = 6$, $A'_{ii} = 6$

(b) $a'_i = Q_{ij} a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $|\mathbf{a}| = (a_i a_i)^{1/2} = \sqrt{6}$, $|\mathbf{a}'| = (a'_i a'_i)^{1/2} = \sqrt{6}$

$A'_{ij} = Q_{ip} Q_{jq} A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, $A_{ii} = 2$, $A'_{ii} = 2$

(c) $a'_i = Q_{ij} a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $|\mathbf{a}| = (a_i a_i)^{1/2} = \sqrt{6}$, $|\mathbf{a}'| = (a'_i a'_i)^{1/2} = \sqrt{6}$

$A'_{ij} = Q_{ip} Q_{jq} A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix}$, $A_{ii} = 6$, $A'_{ii} = 6$

2.11

$\frac{d}{dt}(\mathbf{Q}\mathbf{Q}^T = \mathbf{I}) \Rightarrow \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = 0 \Rightarrow$

$\dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{Q}\dot{\mathbf{Q}}^T = -(\dot{\mathbf{Q}}\mathbf{Q}^T)^T$

2.12

$$Q_{ij} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} = \begin{bmatrix} \cos \theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$b'_i = Q_{ij} b_j = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \cos \theta + b_2 \sin \theta \\ -b_1 \sin \theta + b_2 \cos \theta \end{bmatrix}$$

$$A'_{ij} = Q_{ip} Q_{jq} A_{pq} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^T$$

$$= \begin{bmatrix} A_{11} \cos^2 \theta + (A_{12} + A_{21}) \sin \theta \cos \theta + A_{22} \sin^2 \theta & A_{12} \cos^2 \theta - (A_{11} - A_{22}) \sin \theta \cos \theta - A_{21} \sin^2 \theta \\ A_{21} \cos^2 \theta - (A_{11} - A_{22}) \sin \theta \cos \theta - A_{12} \sin^2 \theta & A_{11} \sin^2 \theta - (A_{12} + A_{21}) \sin \theta \cos \theta + A_{22} \cos^2 \theta \end{bmatrix}$$

2.13

$$a_i = -\frac{1}{2} \varepsilon_{ijk} A_{jk} = -\frac{1}{2} \varepsilon_{ijk} (A_{(jk)} + A_{[jk]}) = -\frac{1}{2} \varepsilon_{ijk} A_{[jk]} \Rightarrow$$

$$a_i \varepsilon_{imn} = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} A_{[jk]} = -\frac{1}{2} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) A_{[jk]} = -\frac{1}{2} (A_{[nm]} - A_{[nm]}) = -A_{[nm]}$$

$$\therefore A_{[nm]} = -a_i \varepsilon_{imn}$$

2.14

Property 7 : Given $A_{ij} B_{ij} = c = \text{scalar}$ and B_{ij} is a second order tensor \Rightarrow

$$A'_{ij} B'_{ij} = c' \Rightarrow A'_{ij} Q_{im} Q_{jn} B_{mn} = c = A_{nm} B_{nm} \Rightarrow A_{nm} = Q_{im} Q_{jn} A'_{ij} \therefore A \text{ is a second order tensor}$$

Property 8 : Given $A_{ij} b_j = c_i$ with b_j and c_i being first order tensors \Rightarrow

$$A'_{ij} b'_j = c'_i \Rightarrow A'_{ij} Q_{jm} b_m = Q_{in} c_n = Q_{in} A_{nm} b_m \Rightarrow A'_{ij} Q_{jm} = Q_{in} A_{nm} \Rightarrow$$

$$Q_{km} (A'_{ij} Q_{jm} = Q_{in} A_{nm}) \Rightarrow A'_{ij} Q_{jm} Q_{km} = Q_{in} A_{nm} Q_{km} \Rightarrow A'_{ij} \delta_{jk} = A'_{ik} = Q_{in} Q_{km} A_{nm}$$

$\therefore A$ is a second order tensor

2.15

$$A_{ij}^{-1} A_{jm} = \frac{1}{2 \det[A]} \varepsilon_{ikl} \varepsilon_{jqr} A_{qk} A_{rl} A_{jm} = \frac{1}{2 \det[A]} \varepsilon_{ikl} \det[A] \varepsilon_{mkl}, \text{ where we have used (2.5.10)}$$

$$= \frac{1}{2} \varepsilon_{ikl} \varepsilon_{mkl} = \frac{1}{2} \varepsilon_{kli} \varepsilon_{klm} = \frac{1}{2} 2 \delta_{im} = \delta_{im}, \text{ where we have used (2.5.7)}$$

2.16

$$I_1 = \text{tr}\mathbf{A} = I_A$$

$$I_2 = \text{tr}\mathbf{A}^2 = A_{ij}A_{ji}$$

$$\text{From (2.11.3) } II_A = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) \Rightarrow A_{ij}A_{ij} = A_{ij}A_{ji} = A_{ii}A_{jj} - 2II_A$$

$$\therefore I_2 = I_A^2 - 2II_A$$

$$I_3 = \text{tr}\mathbf{A}^3,$$

$$\text{Using CT Theorem (2.13.1)} \Rightarrow \mathbf{A}^3 = I_A\mathbf{A}^2 - II_A\mathbf{A} + III_A\mathbf{I}$$

$$\text{Taking trace operation gives } \text{tr}\mathbf{A}^3 = I_A(I_A^2 - 2II_A) - II_AI_A + 3III_A$$

$$\therefore I_3 = I_A^3 - 3II_AI_A + 3III_A$$

2.17

$$\text{Given } A_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$I_A = A_{ii} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_A = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$III_A = \det[A_{ij}] = \lambda_1\lambda_2\lambda_3$$

2.18

$$(a) \quad A_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_A = A_{ii} = -4, \quad II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = 3, \quad III_A = \det[A_{ij}] = 0$$

Characteristic Eqn. $-\lambda^3 - 4\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda + 1)(\lambda + 3) = 0, \lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -3$

$$\underline{\lambda_1 = 0}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$-2n_1^{(1)} + n_2^{(1)} = 0$$

$$n_1^{(1)} - 2n_2^{(1)} = 0$$

$$\therefore n_1^{(1)} = n_2^{(1)} = 0 \text{ and thus } n_3^{(1)} = 1 \Rightarrow \mathbf{n}^{(1)} = \mathbf{e}_3$$

$$\underline{\lambda_2 = -1}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$-n_1^{(2)} + n_2^{(2)} = 0$$

$$n_1^{(2)} - n_2^{(2)} = 0$$

$$n_3^{(2)} = 0$$

$$\therefore n_1^{(2)} = n_2^{(2)} \text{ and thus } \Rightarrow \mathbf{n}^{(2)} = (\mathbf{e}_1 + \mathbf{e}_2) / \sqrt{2}$$

$$\underline{\lambda_3 = -3}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$n_1^{(3)} + n_2^{(3)} = 0$$

$$n_1^{(3)} + n_2^{(3)} = 0$$

$$3n_3^{(3)} = 0$$

$$\therefore n_1^{(3)} = -n_2^{(3)} \text{ and thus } \Rightarrow \mathbf{n}^{(3)} = (\mathbf{e}_1 - \mathbf{e}_2) / \sqrt{2}$$

$$\text{Rotation Matrix to Principal Axes: } Q_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow$$

$$A' = \mathbf{Q} \mathbf{A} \mathbf{Q}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

2.18

$$(b) \quad A_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$I_A = A_{ii} = 0, \quad II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = -4, \quad III_A = \det[A_{ij}] = 0$$

Characteristic Eqn. $-\lambda^3 + 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = 0, \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -2$

$\lambda_1 = 0, (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$$-n_1^{(1)} + n_2^{(1)} = 0$$

$$n_1^{(1)} - n_2^{(1)} = 0$$

$$2n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$$

$\therefore n_1^{(1)} = n_2^{(1)}$ and normalizing $n_1^{(1)} = n_2^{(1)} = 1/\sqrt{2} \Rightarrow \mathbf{n}^{(1)} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$

$\lambda_2 = -2, (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$$n_1^{(2)} + n_2^{(2)} = 0$$

$$n_1^{(2)} + n_2^{(2)} = 0$$

$$4n_3^{(2)} = 0 \Rightarrow n_3^{(2)} = 0$$

$\therefore n_1^{(2)} = -n_2^{(2)}$ and thus $\Rightarrow \mathbf{n}^{(2)} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$

$\lambda_3 = 2, (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$$-3n_1^{(3)} + n_2^{(3)} = 0$$

$$n_1^{(3)} - 3n_2^{(3)} = 0$$

$$0 = 0 \Rightarrow$$

$\therefore n_1^{(3)} = n_2^{(3)} = 0$ and thus $n_3^{(3)} = 1 \Rightarrow \mathbf{n}^{(3)} = \mathbf{e}_3$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$

$$A' = \mathbf{Q}\mathbf{A}\mathbf{Q}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.18

$$(c) \quad A_{ij} = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$I_A = A_{ii} = 18, \quad II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = 99, \quad III_A = \det[A_{ij}] = 162$$

$$\text{Characteristic Eqn. } -\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0 \Rightarrow (\lambda - 9)(\lambda - 6)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 9, \lambda_2 = 6, \lambda_3 = 3$$

$$\underline{\lambda_1 = 9}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$-3n_1^{(1)} - 3n_2^{(1)} = 0$$

$$-3n_1^{(1)} - 3n_2^{(1)} = 0$$

$$-3n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$$

$$\therefore n_1^{(1)} = -n_2^{(1)} \text{ and normalizing } \mathbf{n}^{(1)} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$$

$$\underline{\lambda_2 = 6}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$-3n_2^{(2)} = 0 \Rightarrow n_2^{(2)} = 0$$

$$-3n_1^{(2)} = 0 \Rightarrow n_1^{(2)} = 0$$

$$0 = 0$$

$$\therefore n_3^{(2)} = 1 \text{ and thus } \Rightarrow \mathbf{n}^{(2)} = \mathbf{e}_3$$

$$\underline{\lambda_3 = 3}, \quad (A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$$

$$3n_1^{(3)} - 3n_2^{(3)} = 0$$

$$-3n_1^{(3)} + 3n_2^{(3)} = 0$$

$$3n_3^{(3)} = 0 \Rightarrow n_3^{(3)} = 0$$

$$\therefore n_1^{(3)} = n_2^{(3)} \text{ and normalizing } n_1^{(3)} = n_2^{(3)} = 1/\sqrt{2} \Rightarrow \mathbf{n}^{(3)} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$$

$$\text{Rotation Matrix to Principal Axes: } Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow$$

$$A' = \mathbf{QAQ}^T = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2.19

Multiply basic equation $\mathbf{A}\mathbf{n} = \lambda\mathbf{n}$ by \mathbf{A}^{-1} to get

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{n} = \mathbf{A}^{-1}\lambda\mathbf{n} \Rightarrow \mathbf{n} = \lambda\mathbf{A}^{-1}\mathbf{n} \Rightarrow \mathbf{A}^{-1}\mathbf{n} = \frac{1}{\lambda}\mathbf{n}$$

$\therefore 1/\lambda$ are the principal values of \mathbf{A}^{-1} , and \mathbf{n} are the principal directions

Using principal coordinates,

$$I_A = \lambda_1 + \lambda_2 + \lambda_3, II_A = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, III_A = \lambda_1\lambda_2\lambda_3 \Rightarrow$$

$$II_A / III_A = \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = I_{A^{-1}}$$

2.20

Using principal axes $A_{ij}b_i b_j = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2$

If $\lambda_i \geq 0$, then clearly $A_{ij}b_i b_j \geq 0$

If $A_{ij}b_i b_j \geq 0 \Rightarrow \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 \geq 0$

Since vector \mathbf{b} is arbitrary, choose $\mathbf{b} = (1,0,0) \Rightarrow \lambda_1 \geq 0$

Likewise for other choices $\mathbf{b} = (0,1,0) \Rightarrow \lambda_2 \geq 0$, $\mathbf{b} = (0,0,1) \Rightarrow \lambda_3 \geq 0$

2.21

$$(a) A_{ij} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A_{mm} = 9$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 3\delta_{ij} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$(b) A_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A_{mm} = 3$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$(c) A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix} \Rightarrow A_{mm} = 6$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 2\delta_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -2 & 2 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

2.22

$$\text{From (2.13.2)} \Rightarrow \mathbf{A}^3 = I_A \mathbf{A}^2 - II_A \mathbf{A} + III_A \mathbf{I}$$

$$\text{Multiply by } \mathbf{A} \Rightarrow \mathbf{A}^4 = I_A \mathbf{A}^3 - II_A \mathbf{A}^2 + III_A \mathbf{A}; \text{ substituting in for } \mathbf{A}^3 \text{ from (2.13.2)} \Rightarrow$$

$$\mathbf{A}^4 = I_A (I_A \mathbf{A}^2 - II_A \mathbf{A} + III_A \mathbf{I}) - II_A \mathbf{A}^2 + III_A \mathbf{A} = (I_A^2 - II_A) \mathbf{A}^2 + (III_A - I_A II_A) \mathbf{A} + I_A III_A \mathbf{I}$$

$$\text{Next multiplying by } \mathbf{A} \text{ again} \Rightarrow$$

$$\mathbf{A}^5 = (I_A^2 - II_A) \mathbf{A}^3 + (III_A - I_A II_A) \mathbf{A}^2 + I_A III_A \mathbf{A}; \text{ substituting in for } \mathbf{A}^3 \text{ from (2.13.2)} \Rightarrow$$

$$\mathbf{A}^5 = (I_A^2 - II_A) (I_A \mathbf{A}^2 - II_A \mathbf{A} + III_A \mathbf{I}) + (III_A - I_A II_A) \mathbf{A}^2 + I_A III_A \mathbf{A}$$

$$\mathbf{A}^5 = (I_A^3 - 2I_A II_A + III_A) \mathbf{A}^2 + (I_A III_A - I_A^2 II_A + II_A^2) \mathbf{A} + (I_A^2 III_A - II_A III_A) \mathbf{I}$$

2.23

$$\mathbf{A} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \Rightarrow$$

$$\mathbf{A}^T \mathbf{A} = \mathbf{U}^T \mathbf{R}^T \mathbf{R} \mathbf{U} = \mathbf{U}^T \mathbf{I} \mathbf{U} = \mathbf{U}^T \mathbf{U} = \mathbf{U}^T \mathbf{U} = \mathbf{U}^2$$

$$\mathbf{A} \mathbf{A}^T = \mathbf{V} \mathbf{R} \mathbf{R}^T \mathbf{V}^T = \mathbf{V} \mathbf{I} \mathbf{V}^T = \mathbf{V} \mathbf{V}^T = \mathbf{V} \mathbf{V} = \mathbf{V}^2$$

2.24

(a) $\mathbf{u} = x_1\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + 2x_1x_2x_3\mathbf{e}_3$

$$\nabla \cdot \mathbf{u} = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1x_2$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1x_3\mathbf{e}_1 - 2x_2x_3\mathbf{e}_2 + x_2\mathbf{e}_3$$

$$\nabla^2 \mathbf{u} = \nabla^2 u_1 \mathbf{e}_1 + \nabla^2 u_2 \mathbf{e}_2 + \nabla^2 u_3 \mathbf{e}_3 = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 = 0$$

$$\nabla \mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix}, \quad \text{tr}(\nabla \mathbf{u}) = 1 + x_1 + 2x_1x_2$$

(b) $\mathbf{u} = x_1^2\mathbf{e}_1 + 2x_1x_2\mathbf{e}_2 + x_3^3\mathbf{e}_3$

$$\nabla \cdot \mathbf{u} = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0\mathbf{e}_1 - 0\mathbf{e}_2 + 2x_2\mathbf{e}_3$$

$$\nabla^2 \mathbf{u} = \nabla^2 u_1 \mathbf{e}_1 + \nabla^2 u_2 \mathbf{e}_2 + \nabla^2 u_3 \mathbf{e}_3 = 2\mathbf{e}_1 + 0\mathbf{e}_2 + 6x_3\mathbf{e}_3 = 0$$

$$\nabla \mathbf{u} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}, \quad \text{tr}(\nabla \mathbf{u}) = 4x_1 + 3x_3^2$$

(c) $\mathbf{u} = x_2^2\mathbf{e}_1 + 2x_2x_3\mathbf{e}_2 + 4x_1^2\mathbf{e}_3$

$$\nabla \cdot \mathbf{u} = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2\mathbf{e}_1 - 8x_1\mathbf{e}_2 - 2x_2\mathbf{e}_3$$

$$\nabla^2 \mathbf{u} = \nabla^2 u_1 \mathbf{e}_1 + \nabla^2 u_2 \mathbf{e}_2 + \nabla^2 u_3 \mathbf{e}_3 = 2\mathbf{e}_1 + 0\mathbf{e}_2 + 8\mathbf{e}_3$$

$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad \text{tr}(\nabla \mathbf{u}) = 2x_3$$

2.25

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\begin{aligned}\nabla^2(\phi\psi) &= (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi = \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k} \\ &= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi \cdot \nabla\psi\end{aligned}$$

$$\nabla \cdot (\phi\mathbf{u}) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi \cdot \mathbf{u} + \phi(\nabla \cdot \mathbf{u})$$

(b)

$$\nabla \times (\phi\mathbf{u}) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_{k,j} = \nabla\phi \times \mathbf{u} + \phi(\nabla \times \mathbf{u})$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\varepsilon_{ijk}u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i} v_k) = v_k \varepsilon_{ijk}u_{j,i} + u_j \varepsilon_{ijk}v_{k,i} = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

$$\nabla \times \nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0 \text{ because of symmetry and antisymmetry in } jk$$

$$\nabla \cdot \nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c)

$$\nabla \cdot (\nabla \times \mathbf{u}) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0, \text{ because of symmetry and antisymmetry in } ij$$

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{u}) &= \varepsilon_{nmi}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn} \\ &= \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times (\nabla \times \mathbf{u}) &= \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_j u_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_n u_{n,i} - u_m u_{i,m} \\ &= \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla\mathbf{u}\end{aligned}$$

2.26

Cylindrical coordinates : $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \Rightarrow h_1 = 1, h_2 = r, h_3 = 1$$

$$\hat{\mathbf{e}}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \hat{\mathbf{e}}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \hat{\mathbf{e}}_z = \mathbf{e}_3$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r, \frac{\partial \hat{\mathbf{e}}_r}{\partial r} = \frac{\partial \hat{\mathbf{e}}_\theta}{\partial r} = \frac{\partial \hat{\mathbf{e}}_z}{\partial r} = \frac{\partial \hat{\mathbf{e}}_z}{\partial \theta} = \frac{\partial \hat{\mathbf{e}}_z}{\partial z} = 0$$

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{\mathbf{e}}_r \frac{\partial f}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r}(ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times \mathbf{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{\mathbf{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{\mathbf{e}}_z$$