Solutions

1.1 Numbers 101: The Very Basics

1. (a) The claim makes sense and is true.
   (b) The claim makes no sense; $\sqrt{8}$ isn't a subset.
   (c) The claim makes sense and is true.
   (d) The claim makes sense but is false; consider $a = 0$ and $b = \sqrt{2}$.
   (e) The claim makes sense and is true.
   (f) The claim makes sense but is false: consider $a = 0$.

2. $\mathbb{Z}$ satisfies all the field requirements *except* the one about multiplicative inverses for nonzero elements. The only integers with multiplicative inverses are 1 and -1.

3. The number $1/a$ is an integer only if $a = \pm 1$. The number $1/a$ is rational for all nonzero integers a. The equation $1/a = a$ holds only if $a = \pm 1$.

4. (a) $1 \in S_1$ but $-1 \notin S_1$
   (b) $2 \in S_2$ but $1/2 \notin S_2$
   (c) $\sqrt{2} \in S_3$ but $1/\sqrt{2} = \sqrt{2}/2 \notin S_3$

5. Theorem 1.1 is useful. Because $\mathbb{Q}$ is closed under addition, multiplication, and division (if denominators aren't zero), expressions like those in (a) and (b) are rational—unless, for (b), $p = q = 0$. Expressions involving square roots are different. If $p = q = 1$, for instance, then $\sqrt{p^2 + q^2} = \sqrt{2}$ is irrational; the same expression is rational if $p = 3$ and $q = 4$. The quantity $\sqrt{p^2 + 2pq + q^2}$ is always rational, since $\sqrt{p^2 + 2pq + q^2} = \sqrt{(p+q)^2} = \pm(p+q)$. (Note that $\sqrt{(p+q)^2} = p + q$ may be false.)

6. (a) The answer is no. For instance, the set $\{1, 1/2, 1/3, \ldots\}$ is a subset of $\mathbb{Q}$, but has no least element.
   (b) The set $R = \{1, 10, 100, 1000, \ldots\}$ does have the well-ordering property; every nonempty subset includes a *smallest* power of 10.
   (c) The set $T = \{-3, -2, -1, \ldots, 41, 42\}$ (like all finite sets of real numbers) *does* have the well-ordering property, since every nonempty subset of T is also finite, and hence has a least element.
   (d) If we trade "least" for "greatest" in the well-ordering property, the result no longer holds for $\mathbb{N}$, since $\mathbb{N}$ itself has no greatest element. The property *does* hold for the finite set T, and also for $\mathbb{Z} \setminus \mathbb{N} = \{\ldots, -3, -2, -1, 0\}$.

7. Since $\ln \ln \ln n$ tends to infinity, it must exceed two for large n. The well-ordering property guarantees that a smallest such n_0 exists. (Using a calculator we can see that n_0 has about 703 decimal digits.)

8. All of xr, $x + r$, $x - r$ and x/r can be either rational or irrational if both x and r are irrational. Examples are easy to find.

9. Assume toward contradiction that $\sqrt{3} = a/b$ for integers a and b, and the fraction is written in reduced form. Then squaring both sides gives $3b^2 = a^2$. This implies (essentially as in the proof of Theorem 1.2) that 3 divides both a and b, which contradicts the assumption that a/b is in reduced form.

10. (a) We argue by contradiction: If x is rational, then (by Theorem 1.4) x^2 is rational, too, which contradicts our assumption.
    (b) Another proof by contradiction. If $x = \sqrt{2} + \sqrt{3}$ is rational, then $x^2 = 5 + 2\sqrt{6}$ is rational, too. This implies, in turn, that $\sqrt{6}$ is rational, which is absurd.

(c) Yet another proof by contradiction. Let's write $x = \sqrt{2} + \sqrt{3} + \sqrt{5}$, then we have $x - \sqrt{5} = \sqrt{2} + \sqrt{3}$, and suppose x is rational. Squaring both sides of the last equation and simplifying gives

$$x^2 + 10x\sqrt{5} = 2\sqrt{6},$$

which is progress, since only *two* square roots remain. Squaring again gives

$$x^4 + 20x^3\sqrt{5} + 500x^2 = 24,$$

which is even better, as only *one* square root is left. The last equation implies that

$$\sqrt{5} = 500x^2 = \frac{24 - 500x^2 - x^4}{20x^3}.$$

Because x is rational, so is the right-hand side above. This absurdity completes the proof.

11. Parts (i) and (ii) follow from the fact that $1 < a/b < 2$. For part (iii), note that

$$\frac{a'^2}{b'^2} = \frac{(2b-a)^2}{(a-b)^2} = \frac{4b^2 - 4ab + a^2}{a^2 - 2ab + b^2},$$

and substituting $a^2 = 2b^2$ shows that the last fraction is 2.

This all shows that if $\sqrt{2} = a/b$ holds for *any* positive integers a and b, then we can find a new fraction a'/b' with $\sqrt{2} = a'/b'$ and $b' < b$, which is absurd.

12. Matrix addition in $M_{2\times 2}$ is commutative, but multiplication is not; examples are easy to find. Every matrix A in $M_{2\times 2}$ has an additive inverse $-A$, but multiplicative inverses exist only for some nonzero matrices (those with nonzero determinant); again, examples are easy to find. Distributivity does indeed hold in $M_{2\times 2}$.

13. If a and b are rational, then

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 + 2b^2}.$$

This shows that elements of F have multiplicative inverses in F. The rest is easier.

14. (a) $\mathbb{Z}_2$ is not closed under addition: $1 + 1 = 2 \notin \mathbb{Z}_2$.

(b) $\mathbb{Z}_2$ satisfies all the requirements in Theorem 1.1.

1.2 Sets 101: Getting Started

1. (a) $D \subset I$; $D \in C$.

(b) $B = \{m \in A \mid m \text{ has 31 days}\}$.

(c) $A \times D$ is the set of ordered pairs (January, 2), (February, 2), ..., (December, 2), (January, 3), (February, 3), ..., (December, 3). There are 24 such pairs.

(d) $A \setminus B = \{$February, April, June, September, November$\}$; $B \setminus A = \emptyset$; $A \cap C = \{$November$\}$; $B \cap A = B$; $D \cap I = D$; $D \cup I = I$.

2. (a) $S = \{0, -1\}$; T is the interval of numbers between $(-1 - \sqrt{21})/2 \approx -2.791$ and $(-1 + \sqrt{21})/2 \approx 1.791$.

(b) Decide whether each of the following statements is true or false, and explain: $S \subset \mathbb{N}$ is false because $-1 \notin \mathbb{N}$; $S \subset T$ is true; $T \cap \mathbb{Q} \neq \emptyset$ is true, since $0 \in T \cap \mathbb{Q}$; $-2.8 \in \mathbb{Q} \setminus T$ is true.

(c) The quadratic formula shows that $U = \left\{x \in \mathbb{R} \mid x^2 + x < 0\right\} = (-1, 0)$.

3. (a) Claim (i) is false. As one example, take $A = \mathbb{R}$ and $B = \emptyset$. Then $R \setminus (A \cup B) = \emptyset$ but $(R \setminus A) \cup (R \setminus B) = \mathbb{R}$.

(b) If $A = B$, then both claims are true.

4. Many possibilities exist for (a), (b), and (c). For (d) we could use $I = (0, \infty)$ and $J = (1, \infty)$; note that here, as in *all* possibilities for (d), one interval is contained in the other.

6. No. Here's one way to see why. Consider any set S with 123456789 points; we'll show that it isn't an interval. Because S is finite, it has a smallest element, say s_1, and a second-smallest element, s_2. This means that *no* number between s_1 and s_2 lies in S, and so S is certainly not an interval. (If S were an interval it would contain *all* numbers between s_1 and s_2.)

   (Note that there is nothing special about the number 123456789—*no* finite set can be an interval.)

7. (a) If $S = \{1, 2, 3\}$ then $P(S) = \{\, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\,\}$, a set with 8 members.
   (b) Every subset of S is clearly a subset of T.
   (c) Each subset A of N_{10} corresponds to *two* subsets of N_{11}: A and $A \cup \{11\}$.

8. (a) S has $10 \times 9 \times 8 = 720$ elements.
   (b) T has $10 \times 10 \times 10 = 1000$ elements.
   (c) S_{10} has $10! = 10 \times 9 \times 8 \times \cdots \times 1 = 3628800$ elements.
   (d) The sets N_{10}, S, T, S_{10} have no elements in common.

9. S' has n elements. Each element of S' is formed by omitting one element of S.

10. We have $S \in A_{42}$ if and only if $N_{100} \setminus S \in A_{58}$, so there is a one-to-one correspondence between A_{42} and A_{58}.

11. (a) Three vertical lines.
   (b) Three horizontal lines.
   (c) The set of integer "lattice points" above the x-axis $\mathbb{Z} \times \mathbb{N}$.
   (d) The parabola $y = x^2$.
   (e) A vertical sine curve.
   (f) The empty set.

12. (a) The picture is a diagonal stripe from upper left to lower right.
   (b) The black squares can be described by the set $\{(x, y) \mid x + y \text{ is even}\}$.
   (c) The element $(2, 3, \text{black})$ corresponds to a black square at position $(2, 3)$. The set $G \times \{\text{black}, \text{white}\}$ represents all possible ways to choose a square and color it.
   (d) A picture is, in effect, a *subset* of squares to be colored black. Thus $P(G)$ corresponds to the full set of possible pictures.

1.3 Sets 102: The Idea of a Function

1. There are many possibilities; following are some.

   (a) Let A be the set of all humans who have ever lived, and B the set of all women who have ever lived. The function is not injective, because siblings have the same mother. The function is not surjective, either, since some women are not mothers.
   (b) Let A be the set of all mothers of sons, and B the set of all male humans. Then FIRSTBORNSON : $A \to B$ is one-to-one but not onto.
   (c) Let A be the set of all humans and B the set of all colors. Then EYECOLOR : $A \to B$ is neither one-to-one nor onto, since several people have blue eyes, and nobody has silver eyes.
   (d) Let A be the set of all US citizens and $B = \{\text{January } 1, \text{January } 2, \ldots, \text{December } 31\}$. Then BIRTHDAY : $A \to B$ is onto (every day is someone's birthday) but not one-to-one (several people have the same birthday).

2. (a) The natural domain is all of $\mathbb{R}$.
   (b) The natural domain is all of $\mathbb{R}$ except for points x at which $\cos x = 0$.