Chapter 2 problem solutions

2.3.1

(i) From Eq. (2.17) in the book and the following text, we have the expression $d_s=\lambda z_o/s$ for the separation of the fringes. Here z_o = 10cm and s = 5nm. We can calculate λ using the de Broglie formula $\lambda = h/p$. Remembering also that $E = p^2/2m$ for a free particle, we have

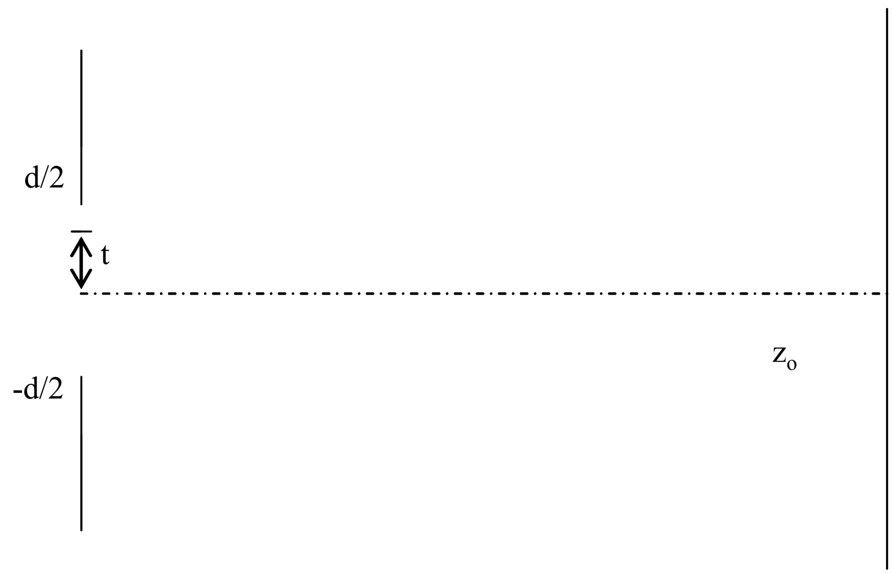
$$p = \sqrt{2mE} = \sqrt{2\cdot(9.11\times10^{-31}\text{kg})\cdot(1\text{eV})\cdot(1.6\times10^{-19}\,\text{J/eV})} = 5.4\text{x}10^{-25}\ \text{kg/s}$$

$$d_s = \lambda z_o / s = hz_o / sp = [(6.626\times10^{-34}\text{J}\cdot\text{s})\times(1\times10^{-1}\text{m})]/[(5\times10^{-9}\,\text{m})\times(5.4\times10^{-25})] = 25\text{mm}$$

(ii) The mass of the proton is 1836 times larger than the mass of the electron. Since p is proportional to the square root of m and d is inversely proportional to p, then the answer should be $25\text{mm}/\sqrt{1836} = 583\mu\text{m}$.

2.3.2

(i) The slit is of width d and the screen is z_o away. Each point in the slit is the source of a spherically expanding wave. The wavefunction at a point (x, z_o) on the screen is the sum of the waves from all the points in the slit. Since it is a continuous set of points, our summation limits to an integral.



Taking the center of the slit as the origin, consider a small section of the slit at a distance t from the center having a width dt. To find the wave function at x on the screen, due to all the point sources within this section, we must propagate the expanding waves through the distance between $(0, t)$ and (x, z_o). This distance r is

$$r = \left(z_o^2 + (x-t)^2\right)^{(1/2)}$$

The wave function at (x, z_o) due to the section dt is

$$d\psi(x, z_o) = \frac{1}{r}\exp(ikr)dt$$

The paraxial approximation that $x << z_o$ implies that $r \sim z_o$. For the $1/r$ term, we can safely assume $1/r \sim 1/z_o$, and neglect that as a constant factor; i.e.,

$$d\psi(x, z_o) = \frac{1}{z_o}\exp(ikr)dt \propto \exp(ikr)dt$$

However, an exponential is very sensitive to small changes in its argument so we cannot assume $r \sim z_o$ in the phase term. The paraxial approximation $x << z_o$ allows us to expand the square root in a Taylor series keeping only first order terms in $(x - t)^2$. Also using $t < x << z_o$

$$r = \left(z_o^2 + (x-t)^2\right)^{(1/2)} \approx z_o\left[1 + \frac{1}{2}\left(\frac{x^2 + t^2 - 2xt}{z_o^2}\right)\right]$$

$$\approx z_o\left[1 + \frac{1}{2}\left(\frac{x^2 - 2xt}{z_o^2}\right)\right] = \left[z_o + \frac{x^2}{2z_o} - \frac{xt}{z_o}\right]$$

Hence,

$$d\psi(x,z_o) \propto \exp(ikr)dt = \exp\left(ik\left(z_o + \frac{x^2}{2z_o}\right)\right)\exp\left(-ik\frac{xt}{z_o}\right)dt$$

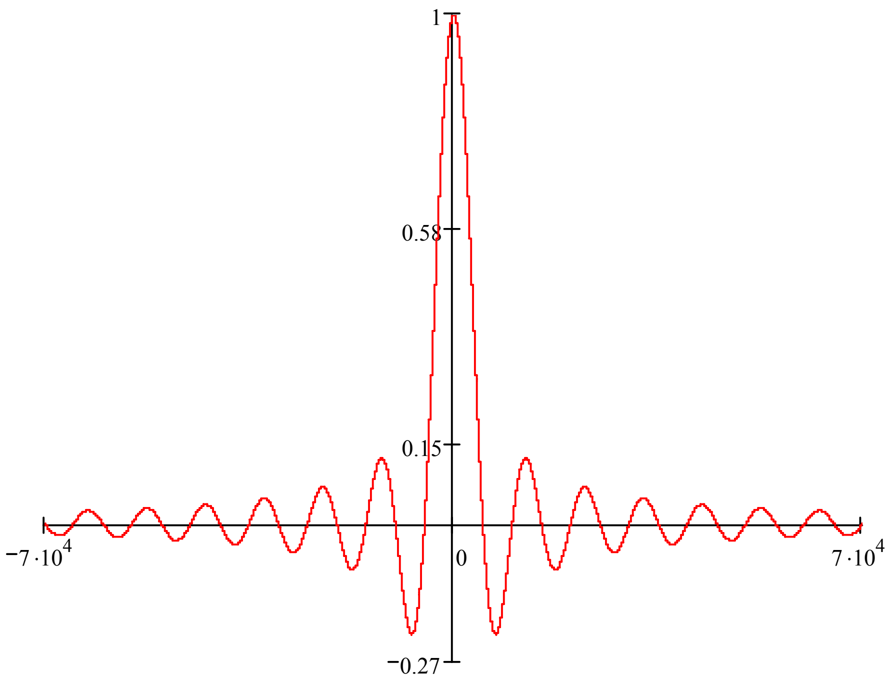
Integrating this over t we get the contribution from the entire slit

$$\psi(x,z_o) = \exp\left(ik\phi\right)\int_{-d/2}^{d/2}\exp\left(-ik\frac{xt}{z_o}\right)dt$$

where
$$\phi = \left(z_o + \frac{x^2}{2z_o}\right)$$

$$\psi(x,z_o) = \exp\left(ik\phi\right)\frac{2z_o}{kx}\sin\left(\frac{kxd}{2z_o}\right) = d\exp(i\phi)\frac{\sin\left(\frac{kxd}{2z_o}\right)}{\left(\frac{kxd}{2z_o}\right)}$$

We have a function of the form $\left(\sin x\right)/x$, also known as the sinc function. Below is a plot of $\psi(x,z_o)$.



distance on screen (microns)

The function crosses zero each time the argument of the sine is a non-zero multiple of π, so the widths of each of the side fringes is $\frac{\lambda z_o}{d}$. More importantly, the width of the bright central lobe is $2\frac{\lambda z_o}{d}$, which says the electrons diffract more as

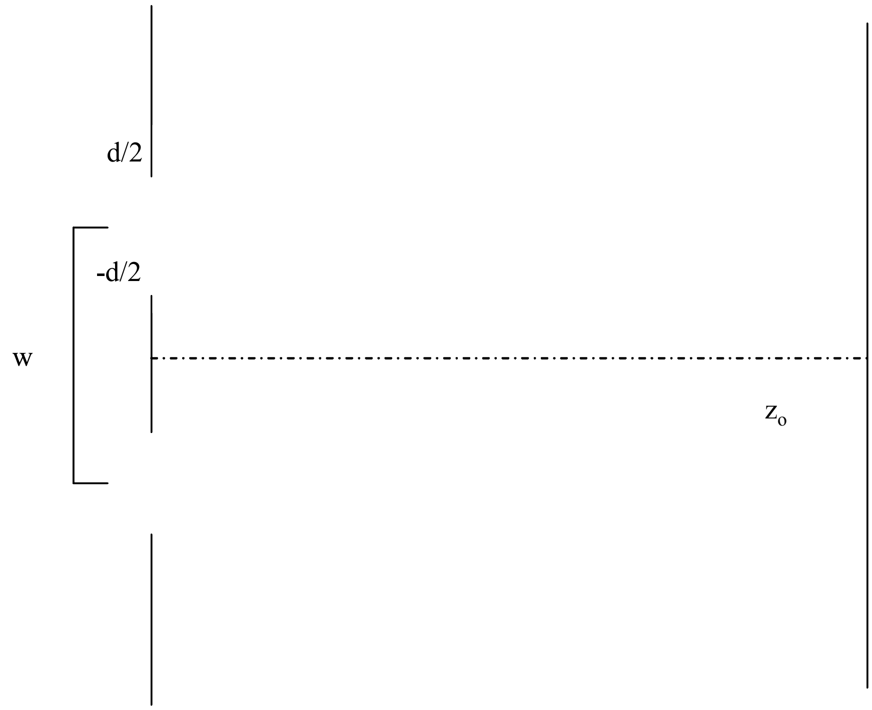
(a) the slit gets narrower for given λ

(b) the wavelength gets larger for a given slit width

The z_o term just says the farther the screen, the larger the spot.

(ii) The intensity of light at a point on the phosphorescent screen is proportional to the probability density of the electron at that point

$$I(x,z_o) = |\psi(x,z_o)|^2 = \mathrm{sinc}^2\left(\frac{kxd}{2z_o}\right)$$



(iii) Two finite slits

For two slits we can essentially use the technique in part (i). We integrate over each slit separately and add the results. Let the slit separation (center to center) be w and the variable of integration t measured from the center of each slit. The limits of integration are $t = -d/2$ to $t = d/2$.

For the top slit this results in

$$r = \left(z_o^2 + (x - \frac{w}{2} - t)^2\right)^{(1/2)} \approx z_o\left[1 + \frac{1}{2}\left(\frac{(x-\frac{w}{2})^2 + t^2 - 2(x-\frac{w}{2})t}{z_o^2}\right)\right]$$

$$\approx z_o\left[1 + \frac{1}{2}\left(\frac{(x-w/2)^2 - 2(x-w/2)t}{z_o^2}\right)\right] = \left[z_o + \frac{(x-w/2)^2}{2z_o} - \frac{(x-w/2)t}{z_o}\right]$$

Note that this is the same form as we had for the single slit just with x shifted to $x - w/2$. For the bottom slit the shift is $x \to x + w/2$ i.e.,

$$d\psi(x,z_o) = \exp\left(ik\left(z_o + \frac{(x-w/2)^2}{2z_o}\right)\right)\exp\left(-ik\frac{(x-w/2)t}{z_o}\right)dt \qquad \text{from the top slit}$$

$$d\psi(x,z_o)=\exp\left(ik\left(z_o+\frac{(x+w/2)^2}{2z_o}\right)\right)\exp\left(-ik\frac{(x+w/2)t}{z_o}\right)dt \quad \text{from the bottom slit}$$

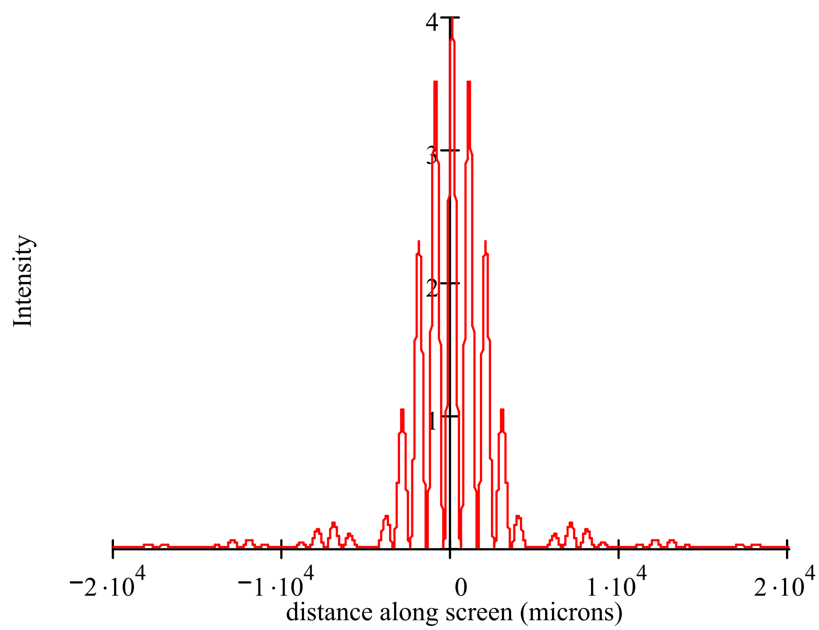
Integrating each and adding gives

$$d\exp\left(ik\left(z_o+\frac{(x-w/2)^2}{2z_o}\right)\right)\frac{\sin\left(\frac{k(x-w/2)d}{2z_o}\right)}{\left(\frac{k(x-w/2)d}{2z_o}\right)}+d\exp\left(ik\left(z_o+\frac{(x+w/2)^2}{2z_o}\right)\right)\frac{\sin\left(\frac{k(x+w/2)d}{2z_o}\right)}{\left(\frac{k(x+w/2)d}{2z_o}\right)}$$

Ignoring constant phase factors and constants we have

$$\exp\left(ik\frac{(x-w/2)^2}{2z_o}\right)\frac{\sin\left(\frac{k(x-w/2)d}{2z_o}\right)}{\left(\frac{k(x-w/2)d}{2z_o}\right)}+\exp\left(ik\frac{(x+w/2)^2}{2z_o}\right)\frac{\sin\left(\frac{k(x+w/2)d}{2z_o}\right)}{\left(\frac{k(x+w/2)d}{2z_o}\right)}$$

Plotting the modulus squared of this we get



Note: Given the numbers in the problem w^2/z_0^2 was very small so neglecting this will still give essentially the correct plot. However, in general it may not be the case that the slit spacing is negligible.

2.4.1

We will use the following test for linearity for a non-zero solution $\psi(x)$.

If $\psi(x)$ is a solution then $a\psi(x)$ is also a solution where a is an arbitrary constant.

If we substitute $a\psi(x)$ into each of the equations we get the following.

i) $az\dfrac{d\psi}{dz}+ag(z)\psi(z)=0$

We see that the a's will cancel, meaning that therefore $a\psi(x)$ is also a solution, so this equation is **LINEAR.**

ii) $a^2\psi(z)\dfrac{d\psi(z)}{dz}+a\psi(z)=0$

The constant a cannot be canceled in this equation, so it is **NOT LINEAR. (**In other words, given that $\psi(x)$ is a solution of $\psi(z)\dfrac{d\psi(z)}{dz}+\psi(z)=0$, the only value of a for which $a\psi(x)$ is also a solution is $a = 1$.)

iii) $a\dfrac{d^2\psi(z)}{dz^2}+ab\dfrac{d\psi(z)}{dz}=ac\psi(z)$

We see that the a's will cancel, so this is **LINEAR.**

iv) $a\dfrac{d^3\psi(z)}{dz^3}=1$

The constant a cannot be canceled in this equation, so it is **NOT LINEAR.**

v) $a\dfrac{d^2\psi(z)}{dz^2}+a(1+a^2\left|\psi(z)^2\right|)\dfrac{d\psi(z)}{dz}=ag\psi(z)$

The constant a cannot be canceled in this equation, so it is **NOT LINEAR.**

2.6.1

The normalized wavefunctions for the various different levels in the potential well are

$$\psi_n(z) = \sqrt{\frac{2}{L_z}} \sin\left(\frac{n\pi z}{L_z}\right)$$

The lowest energy state is $n = 1$, and we are given $L_z = 1$ nm.

The probability of finding the electron between 0.1 and 0.2 nm from one side of the well is, using nanometer units for distance,

$$\begin{aligned}
P &= \int_{0.1}^{0.2} |\psi_1(z)|^2 \, dz = \int_{0.1}^{0.2} 2\sin^2(\pi z) dz \\
&= \int_{0.1}^{0.2} [1 - \cos(2\pi z)] dz \\
&= 0.1 - \int_{0.1}^{0.2} \cos(2\pi z) dz \\
&= 0.1 - \frac{1}{2\pi}[\sin(2\pi \times 0.2) - \sin(2\pi \times 0.1)] \\
&= 0.042
\end{aligned}$$

(Note: For computation purposes, remember that the argument of the sine is in radians and not degrees. For example, when we say $\sin(\pi) = 0$, it is implicit here that we mean π radians.)

2.6.2

(i) Odd, since

$$f(x) = \sin(x)$$
$$f(-x) = \sin(-x) = -\sin(x) = -f(x)$$

(ii) Neither even nor odd, since

$$f(x) = \exp(ix) = \cos(x) + i\sin(x)$$
$$f(-x) = \exp(-ix) = \cos(x) - i\sin(x)$$

(iii) Even, since

$$f(x) = (x-a)(x+a) = x^2 - a^2$$
$$f(-x) = x^2 - a^2 = f(x)$$

(iv) Even, since

$$f(x) = \exp(ix) + \exp(-ix) = 2\cos(x)$$
$$f(-x) = 2\cos(-x) = 2\cos(x) = f(x)$$

(v) Odd, since

$$f(x) = x(x^2 - 1)$$
$$f(-x) = -x(x^2 - 1) = -f(x)$$

2.6.3

(i) $\sin\left(7\pi z / L_z\right)$

Yes (this is the solution for $n = 7$ for such a simple well)

(ii) $\cos\left(2\pi z / L_z\right)$

No (this does not fit the boundary conditions at the walls of the well, not being zero amplitude at the walls)

(iii) $0.5\sin\left(3\pi z / L_z\right) + 0.2\sin\left(\pi z / L_z\right)$

No (this is a superposition of two eigenfunctions, but that is not a solution of the time-independent Schrödinger equation)

(iv) $\exp(-0.4i)\sin\left(2\pi z / L_z\right)$

Yes (This is the solution for $n = 2$, with a complex factor that makes no difference in the time-independent Schrödinger equation.)

2.6.4

We have an infinite potential well of width L_z in each of the three dimensions.

(i) Given the note at the end of the problem, we can write the solution as a product of three solutions

$$\psi_{n_x n_y n_z}(x,y,z)=\sqrt{\frac{2}{L_z}}\sin\left(\frac{n_x\pi x}{L_z}\right)\times\sqrt{\frac{2}{L_z}}\sin\left(\frac{n_y\pi y}{L_z}\right)\times\sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi z}{L_z}\right)$$

With the allowed energies of the three dimensional system being

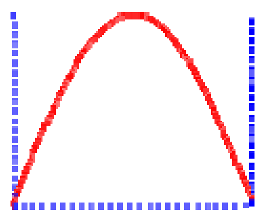
$$E_n=E_{n_x}+E_{n_y}+E_{n_z}=E_1^\infty(n_x^2+n_y^2+n_z^2)$$

where

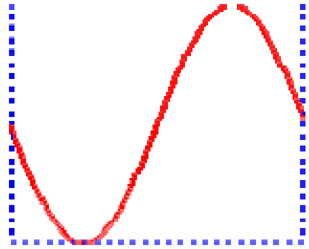
$$E_1^\infty=\frac{\hbar^2}{2m}\left(\frac{\pi}{L_z}\right)^2$$

is the lowest allowed energy of a particle in a one-dimensional infinite potential well.

(ii) The lowest allowed state corresponds to $(n_x,n_y,n_z)=(1,1,1)$ and has an energy $E_1=3E_1^\infty$. In this state the wave function in each of the three directions looks like



The next three states all have the same energy $E_{2,3,4}=6E_1^\infty$ and correspond to $(n_x,n_y,n_z)=(2,1,1)$, $(n_x,n_y,n_z)=(1,2,1)$, $(n_x,n_y,n_z)=(1,1,2)$. In these three cases, the wave function along two of the dimensions looks as above, but along the third dimension it is the next higher state of the infinite well



(iii) These last three states have equal energy and hence are degenerate (i.e., they have a degeneracy of 3) because the width of the well is the same in all three dimensions.

2.7.1

To check orthogonality on [-1,1] we take the inner product of the two functions. If they are orthogonal on the interval, then the inner product (orthogonality integral) is zero, i.e.,

$$\int_{-1}^{1} f(x)g^*(x)dx = 0$$

Since the interval is symmetric about $x = 0$, we can use the parity of the integrand to determine by inspection whether the integral is zero or not. But we should be careful with periodic functions, as in (v).

(i) orthogonal

$$\int_{-1}^{1} x^3 = 0 \qquad \text{(since } x^3 \text{ is odd)}$$

(ii) not orthogonal

$$\int_{-1}^{1} x^4 \neq 0 \qquad \text{(since } x^4 \text{ is even)}$$

(iii) not orthogonal

$$\int_{-1}^{1} x \sin x dx \neq 0 \qquad \text{(since } x \sin x \text{ is even)}$$

(iv) not orthogonal

$$\int_{-1}^{1} x \exp\left(\frac{-i\pi x}{2}\right) dx = \int_{-1}^{1}\left(x\cos\frac{\pi x}{2} - ix\sin\frac{\pi x}{2}\right)dx$$

note that $\cos(\pi x/2)$ and $\sin(\pi x/2)$ are periodic with period 4, so the interval is less than a period.

$$0 - i\int_{-1}^{1}\left(x\sin\frac{\pi x}{2}\right)dx \neq 0$$

(since $x\cos\frac{\pi x}{2}$ is odd and $x\sin\frac{\pi x}{2}$ is even)

(v) orthogonal

$$\int_{-1}^{1} \exp(-4\pi i x)dx = \int_{-1}^{1}[\cos(4\pi x) - i\sin(4\pi x)]dx = 0$$

because the period of both $\cos(4\pi x)$ and $\sin(4\pi x)$ is 2, so the interval is an integral number of periods (in this case one full period). The integral of a sinusoid over one period is zero.

2.7.2

(i) $$f_0(x)=1,\quad f_1(x)=x,\quad f_2(x)=x^2\quad \ldots f_n(x)=x^n$$

For a counter example to show these functions are not all orthogonal, we can choose the x and x^3 functions, leading to an orthogonality integral

$$\int_{-1}^{1} x\times x^3 dx=\int_{-1}^{1} x^4 dx\neq 0 \qquad \text{(since } x^4 \text{ is even)}$$

(ii) Let the unnormalized functions be labeled as $h_i(x)$. So we have, for our first member of this set of functions

$$h_0(x)=f_0(x)=1$$

To normalize a function we divide it by the appropriate normalization factor, which is the square root of the integral of its modulus squared over the interval of interest. For this function, the integral of its modulus squared over the interval of interest is

$$\int_{-1}^{1}\left|f_0(x)\right|^2 dx=\int_{-1}^{1} 1dx=2$$

Thus the normalized version is $g_0(x)=\dfrac{1}{\sqrt{2}}$

We can check that $g_0(x)$ is normalized; i.e.,

$$\int_{-1}^{1}\left|g_0(x)\right|^2=1 \qquad (1)$$

Now we try to construct a function that is orthogonal to $g_0(x)$ by constructing a combination of $g_0(x)$ and $f_1(x)$ that is orthogonal to $g_0(x)$. One appropriate way of writing such a combination is $h_1(x)=f_1(x)+a_{10}g_0(x)$. We want this to be orthogonal to $g_0(x)$, so we require

$$\int_{-1}^{1}\left(f_1(x)+a_{10}g_0(x)\right)g_0(x)dx=0$$

so, using Eq. (1) $$\int_{-1}^{1} f_1(x)g_0(x)dx=-a_{10}$$

Hence $-a_{10}=\int_{-1}^{1}\dfrac{x}{\sqrt{2}}dx=0$. Thus, in this particular case, $h_1(x)=f_1(x)=x$.

Normalizing $$\int_{-1}^{1}\left|h_1(x)\right|^2 dx=\int_{-1}^{1} x^2 dx=2/3$$

Hence $$g_1(x)=\sqrt{\frac{3}{2}}x$$

So now we have constructed a function $g_1(x)$ that is normalized and orthogonal to $g_0(x)$.

c) Similarly, we now try to construct a function $h_2(x)$ that is orthogonal to $g_0(x)$ and $g_1(x)$ by adding amounts of these two functions and some of the next linearly independent function in our set, i.e.,

$$h_2(x) = f_2(x) + a_{20}g_0(x) + a_{21}(x)g_1(x)$$

Orthogonalizing to g_0 and g_1 gives

$$a_{21} = -\int_{-1}^{1} x^2 \sqrt{\frac{3}{2}} x dx = 0 \quad \text{and} \quad a_{20} = \int_{-1}^{1} x^2 \sqrt{\frac{1}{2}} dx = -\sqrt{\frac{2}{3}}$$

So
$$h_2(x) = x^2 - \frac{1}{3}$$

Normalizing, we have
$$\int_{-1}^{1} |h_2(x)|^2 dx = \int_{-1}^{1} (x^4 + \frac{1}{9} - \frac{2}{3}x^2) dx = \frac{8}{45}$$

and hence
$$g_2(x) = \frac{1}{3}\sqrt{\frac{45}{8}}(3x^2 - 1) = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1)$$

d) We write

$$h_i(x) = f_i(x) + a_{i0}g_0(x) + a_{i1}g_1(x) + \ldots\ldots + a_{i,i-1}g_{i-1}(x)$$

Orthogonalizing to $g_j(x)$, we have

$$\int_{-1}^{1} h_i(x)g_j(x)dx = 0 \Rightarrow \int_{-1}^{1} f_i(x)g_j(x)dx + a_{ij}\int_{-1}^{1} g_j(x)g_j(x)dx = 0$$

$$\Rightarrow a_{ij} = -\int_{-1}^{1} f_i(x)g_j(x)dx$$

e) For the function $g_3(x)$, we start with $h_3(x) = f_3(x) + a_{30}g_0(x) + a_{31}g_1(x) + a_{32}g_2(x)$

Orthogonalizing to $g_0(x)$, $g_1(x)$, and $g_2(x)$ leads to

$$a_{30} = -\int_{-1}^{1} x^3 \sqrt{\frac{1}{2}} dx = 0 \qquad a_{31} = \int_{-1}^{1} x^3 \sqrt{\frac{3}{2}} x dx = -\frac{2}{5}\sqrt{\frac{3}{2}} \qquad a_{32} = \int_{-1}^{1} x^3 \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1)dx = 0$$

So
$$h_3(x) = x^3 - \frac{2}{5}(\frac{3}{2})x = 5x^3 - 3x$$

Normalizing, we have
$$\int_{-1}^{1} |h_3(x)|^2 dx = \int_{-1}^{1} (25x^6 + 9x^2 - 30x^4) dx = \frac{8}{7}$$

Hence
$$g_3(x) = \frac{1}{2}\sqrt{\frac{7}{2}}(5x^3 - 3x)$$

So finally we have

$$g_0(x) = \sqrt{\frac{1}{2}} \qquad g_1(x) = \sqrt{\frac{3}{2}}x \qquad g_2(x) = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1) \qquad g_3(x) = \frac{1}{2}\sqrt{\frac{7}{2}}(5x^3 - 3x)$$

f) The above is not the only set of orthogonal normalized functions for this interval in powers of x. If we start with a different function in the series we will in general get a new set. For instance, we could make the following choice for our first function

$$h_0 = f_3(x) = x$$

Normalizing this choice gives $$g_0(x) = \sqrt{\frac{3}{2}}x$$

Then for our second function, we would have

$$h_1(x) = f_2(x) + a_{10}g_0(x)$$

Orthogonalizing to $g_0(x)$ leads to $$a_{10} = -\int_{-1}^{1} x^2 \sqrt{\frac{3}{2}} x dx = 0$$

Hence $$h_1(x) = x^2$$

Normalizing this function gives $$g_1(x) = \sqrt{\frac{5}{2}}x^2$$

and so on. Note that these functions g are a different and orthonormal set; for example, we have no function that is simply proportional to x^2 in the set of functions in part (e) above.