

Problem 1.1 An imaging lens in a digital camera has a focal length of 6 cm. How far should the lens be from the camera's CCD array to focus on an object

- (a) 12 cm in front of the lens?
- (b) 15 cm in front of the lens?

Solution:

(a) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 6$ cm and $d_o = 12$ cm, so $d_i = \boxed{12 \text{ cm}}$.

(b) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 6$ cm and $d_o = 15$ cm, so $d_i = \boxed{10 \text{ cm}}$.

Problem 1.2 An imaging lens in a digital camera has a focal length of 4 cm.

How far should the lens be from the camera's CCD array to focus on an object

- (a) 12 cm in front of the lens;
- (b) 8 cm in front of the lens.

Solution:

(a) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 4$ cm and $d_o = 12$ cm, so $d_i = \boxed{6 \text{ cm}}$.

(b) The lens equation (Eq. (1.1)) is $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

Here, $f = 4$ cm and $d_o = 8$ cm, so $d_i = \boxed{8 \text{ cm}}$.

Problem 1.3 The following program loads an image stored in `clown.mat` as $I_o(x, y)$, passes it through an imaging system with the PSF given by Eq. (1.6), and displays $I_o(x, y)$ and $I_i(x, y)$. Parameters Δ , D , d_i , and λ (all in mm) are specified in the program's first line.

```
clear;Delta=0.0002;D=0.03;lambda=0.0000005;di=0.003;T=round(0.01/Delta);
for I=1:T;for J=1:T;x2y2(I,J)=(I-T/2).*(I-T/2)+(J-T/2).*(J-T/2);end;end;
gamma=pi*D/lambda*sqrt(x2y2./(x2y2+di*di/Delta/Delta));
h=2*besselj(1,gamma)./gamma;
h(T/2,T/2)=(h(T/2+1,T/2)+h(T/2-1,T/2)+h(T/2,T/2+1)+h(T/2,T/2-1))/4;
h=h.*h;H=h(T/2-5:T/2+5,T/2-5:T/2+5);load clown.mat;Y=conv2(X,H);
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_o(x, y)$ (input) and $I_i(x, y)$ (output).

Solution: $I_o(x, y)$ is at left and $I_i(x, y)$ is at right.

The image formed by the optical system is blurred, as expected.



Problem 1.4 Compare the azimuth resolution of a real-aperture radar with that of a synthetic-aperture radar, with both pointed at the ground from an aircraft at a range $R = 5$ km. Both systems operate at $\lambda = 3$ cm and utilize a 2-m-long antenna.

Solution: For the real-aperture radar,

$$\Delta Y'_{\min} = \frac{\lambda R}{l_y} = \frac{3 \times 10^{-2} \times 5 \times 10^3}{2} = 75 \text{ m.}$$

For the SAR,

$$\Delta Y'_{\min} = \frac{l_y}{2} = \frac{2}{2} = 1 \text{ m.}$$

Problem 1.5 A 2-m-long antenna is used to form a synthetic-aperture radar from a range of 100 km. What is the length of the synthetic aperture?

Solution: Scaling the range in Fig. 1-21 from 400 km down to 100 km leads to a synthetic aperture shorter by the same factor. Hence, the synthetic aperture is of length $8 \text{ km}/4 = 2 \text{ km}$.

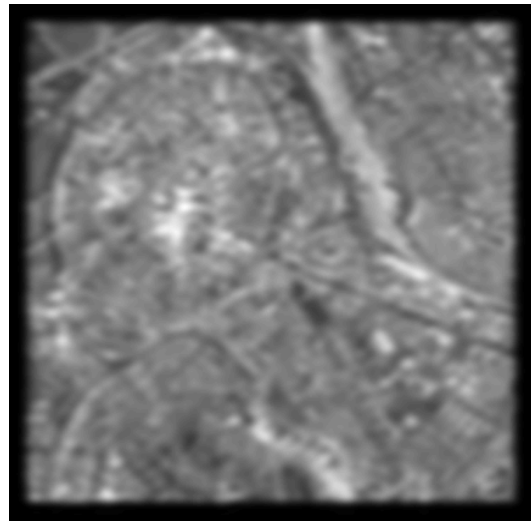
Problem 1.6 The following program loads an image stored in `sar.mat` as $I_o(x, y)$, passes it through an imaging system with the PSF given by Eq. (1.15), and displays $I_o(x, y)$ and $I_i(x, y)$. Parameters Δ , τ , and l are specified in the program's first line.

```
clear;Delta=0.1;l=5;tau=1;I=[-15:15];z=pi*1.8*Delta*I/l;load sar.mat;
hy=sin(pi*z)./(pi*z);hy(16)=1;hy=hy.*hy;hx=exp(-2.77*Delta*Delta*I.*I/tau/tau);
H=hy'*hx;Y=conv2(X,H);
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_o(x, y)$ (input) and $I_i(x, y)$ (output).

Solution: $I_o(x, y)$ is at left and $I_i(x, y)$ is at right.

The image formed by the radar system is blurred, as expected.



Problem 1.7 (This problem assumes prior knowledge of the 1-D Fourier transform (FT)). The basic CT problem is to reconstruct $\alpha(\xi, \eta)$ in Eq. (1.18) from $p(r, \theta)$. One way to do this is as follows:

- (a) Take the FT of Eq. (1.18), transforming r to f . Define $p(-r, \theta) = p(r, \theta + \pi)$.
- (b) Define and substitute $\mu = f \cos \theta$ and $\nu = f \sin \theta$ in this FT.
- (c) Show that the result defines 2 FTs, transforming ξ to μ and η to ν , and that $\mathbf{A}(\mu, \nu) = \mathbf{P}(f, \theta)$. Hence, $\alpha(\xi, \eta)$ is the inverse FT of $\mathbf{P}(f, \theta)$.

Solution:

- (a) The FT of Eq. (1.18) taking r to f is

$$\mathbf{P}(f, \theta) = \mathcal{F}\{p(r, \theta)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(\xi, \eta) e^{-j2\pi f(\xi \cos \theta + \eta \sin \theta)} d\xi d\eta.$$

- (b) Substituting gives

$$\mathbf{P}(f, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(\xi, \eta) e^{-j2\pi\mu\xi} e^{-j2\pi\nu\eta} d\xi d\eta.$$

- (c) $\mathbf{P}(f, \theta) = \mathcal{F}_{\xi \rightarrow \mu} \{ \mathcal{F}_{\eta \rightarrow \nu} \{ \alpha(\xi, \eta) \} \} = \mathbf{A}(\mu, \nu)$,

Problem 1.8 The following program loads an image stored in `mri.mat` as $I_o(x, y)$, passes it through an imaging system with the PSF given by Eq. (1.20), and displays $I_o(x, y)$ and $I_i(x, y)$. Parameters Δ , N , and dk are specified in the program's first line.

```
clear;N=16;Delta=0.01;dk=1;I=[-60:60];load mri.mat;  
h=dk*sin(pi*N*dk*I*Delta)./sin(pi*dk*I*Delta);h(61)=N;H=h'*h;Y=conv2(X,H);  
figure,imagesc(X),axis off,colormap(gray),figure,imagesc(Y),axis off,colormap(gray)
```

Run the program and display $I_o(x, y)$ (input) and $I_i(x, y)$ (output).

Solution: $I_o(x, y)$ is at left and $I_i(x, y)$ is at right.

The image formed by the MRI system is blurred, as expected.



Problem 1.9 This problem shows how beamforming works on a linear array of transducers, as illustrated in Fig. 1-35, in a medium with a wave speed of 1540 m/s. We are given a linear array of transducers located 1.54 cm apart along the x axis, with the n th transducer located at $x = 1.54n$ cm. Outputs $\{y_n(t)\}$ from the transducers are delayed and summed to produce the signal $y(t) = \sum_n y_n(t - 0.05n)$. In what direction (angle from perpendicular to the array) is the array focused?

Solution: Consider a plane wave (impulse in space and time) $\delta(t - x \sin \theta - y \cos \theta)$ arriving at the array from a direction θ (angle from perpendicular to the array). The plane wave hits the n th transducer at $t = n \sin(\theta) \frac{1.54 \text{ cm}}{1540 \text{ m/s}} = 0.1n \sin \theta$. Setting the delay between transducers $0.05n = 0.1n \sin \theta$ gives $\theta = 30^\circ$.

Problem 2.1 Compute the following convolutions:

(a) $e^{-t} u(t) * e^{-2t} u(t)$

(b) $e^{-2t} u(t) * e^{-3t} u(t)$

(c) $e^{-3t} u(t) * e^{-3t} u(t)$

Solution:

The convolution of two causal signals is $y(t) = u(t) \int_0^t h(\tau) x(t - \tau) d\tau$.

(a): $e^{-t} u(t) * e^{-2t} u(t) = u(t) \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau = e^{-2t} u(t) \int_0^t e^{\tau} d\tau = e^{-2t} u(t) [e^t - 1] =$
 $\boxed{e^{-t} u(t) - e^{-2t} u(t)}$

(b): $e^{-2t} u(t) * e^{-3t} u(t) = u(t) \int_0^t e^{-2\tau} e^{-3(t-\tau)} d\tau = e^{-3t} u(t) \int_0^t e^{\tau} d\tau = e^{-3t} u(t) [e^t - 1] =$
 $\boxed{e^{-2t} u(t) - e^{-3t} u(t)}$

(c): $e^{-3t} u(t) * e^{-3t} u(t) = u(t) \int_0^t e^{-3\tau} e^{-3(t-\tau)} d\tau = e^{-3t} u(t) \int_0^t d\tau = \boxed{te^{-3t} u(t)}$

Problem 2.2 Show that the spectrum of $\frac{\sin(20\pi t)}{\pi t} \frac{\sin(10\pi t)}{\pi t}$ is zero for $|f| > 15$ Hz.

Solution:

Using the Fourier transform property $\mathcal{F}[x(t)y(t)] = \mathbf{X}(f) * \mathbf{Y}(f)$,

and the property that the width of a convolution is the sum of the widths, the bandwidth of the product of two signals is the sum of their bandwidths.

$$\mathcal{F}\left[\frac{\sin(20\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 10 \text{ Hz} \\ 0, & |f| > 10 \text{ Hz} \end{cases}$$

and

$$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 5 \text{ Hz} \\ 0, & |f| > 5 \text{ Hz}. \end{cases}$$

Hence,

$$\mathcal{F}\left[\frac{\sin(20\pi t)}{\pi t} \frac{\sin(10\pi t)}{\pi t}\right] = 0 \quad \text{for } |f| > 15 \text{ Hz}.$$

Problem 2.3 Using only Fourier transform properties, show that

$$\frac{\sin(10\pi t)}{\pi t} [1 + 2 \cos(20\pi t)] = \frac{\sin(30\pi t)}{\pi t}.$$

Solution:

$$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t}\right] = \begin{cases} 1, & |f| < 5 \\ 0, & |f| > 5 \end{cases}$$

Using the modulation property:

$$\mathcal{F}[x(t) \cos(2\pi f_0 t)] = \frac{1}{2}\mathbf{X}(f - f_0) + \frac{1}{2}\mathbf{X}(f + f_0),$$

we have

$$\mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t} \cos(20\pi t)\right] = \begin{cases} \frac{1}{2}, & 5 \text{ Hz} < |f| < 15 \text{ Hz}, \\ 0, & \text{otherwise.} \end{cases}$$

Adding the first equation to double the second equation gives

$$\begin{aligned} \mathcal{F}\left[\frac{\sin(10\pi t)}{\pi t} (1 + 2 \cos(20\pi t))\right] &= \begin{cases} 1, & |f| < 5 \text{ Hz} \\ 0, & |f| > 5 \text{ Hz} \end{cases} + \begin{cases} 1, & 5 \text{ Hz} < |f| < 15 \text{ Hz}, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & |f| < 15 \text{ Hz}, \\ 0, & |f| > 15 \text{ Hz}. \end{cases} \end{aligned}$$

The inverse Fourier transform of this result is $\frac{\sin(30\pi t)}{\pi t}$.

The sum of this lowpass filter and bandpass filter is another lowpass filter:

