

## Chapter 1

1. (i)  $A^0 = 4, A^1 = 3, A^2 = 2, A^3 = 1$

$$A_i = \eta_{ik} A^k \Rightarrow A_0 = 4, A_1 = -3, A_2 = -2, A_3 = -1$$

$$A_i A^i = 4^2 - 3^2 - 2^2 - 1^2 = 2 > 0 \Rightarrow A^i \text{ is timelike.}$$

(ii)  $x^2 + y^2 = 1 \Rightarrow x dx + y dy = 0 \Rightarrow dx = -\lambda y, dy = \lambda x$

$$z = 0 \Rightarrow dz = 0, t = 0 \Rightarrow dt = 0.$$

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0 - \lambda^2 x^2 - \lambda^2 y^2 - 0 = -\lambda^2 < 0$$

Hence the tangent vector is spacelike.

(iii)  $\phi \equiv x^2 + y^2 + z^2 - c^2 t^2 = 1.$

$$\text{Normal vector is } \left( \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \equiv (-2c^2 t, 2x, 2y, 2z) = A_i \text{ (say).}$$

$$A^i = -2t, -2x, -2y, -2z$$

$$\Rightarrow A_i A^i = 4c^2 t^2 - 4x^2 - 4y^2 - 4z^2 = -4 < 0.$$

Hence the normal vector is spacelike.

(iv) We have  $\frac{dx^1}{d\lambda} = r \sin\theta, \frac{dx^2}{d\lambda} = r \cos\theta, \frac{dx^3}{d\lambda} = z, \frac{dx^0}{d\lambda} = \sqrt{r^2 + z^2}.$

$$n_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} = -r^2 \sin^2\theta - r^2 \cos^2\theta - z^2 + r^2 + z^2 = 0.$$

$\Rightarrow$  The vector is null.

2. Using the electromagnetic field tensor  $F_{ik}$ , where

$$F_{01} = E_1, F_{02} = E_2, F_{03} = E_3$$

$$F_{32} = B_1, F_{13} = B_2, F_{21} = B_3$$

$\mathbf{E} = (E_1, E_2, E_3), \mathbf{B} = (B_1, B_2, B_3)$  being the electric and magnetic field vectors in 3 dimensions, we use the special Lorentz transformation in the form

$$x'^i = L^i_k x^k$$

with the non-zero components of  $L^i_k$  as  $L^0_0 = \gamma, L^0_1 = -\gamma v, L^1_0 = -\gamma v, L^1_1 = \gamma, \text{ and } L^2_2 = L^3_3 = 1.$

The tensor transformation law gives  $F_{ik} \rightarrow F'_{ik}$ , where

$$F'_{ik} = L_i^m L_k^n F_{mn}.$$

We also have  $L_0^0 = \gamma$ ,  $L_0^1 = \gamma v = L_1^0$ ,  $L_1^1 = \gamma$ ,  $L_2^2 = L_3^3 = 1$ .  
So, for example

$$\begin{aligned} F'_{01} &= L_0^0 L_1^1 F_{01} + L_0^1 L_1^0 F_{10} \\ &= \gamma^2(1 - v^2)F_{01} = F_{01}, \end{aligned}$$

i.e.,  $E'_1 = E_1$ .

It can be easily verified that the other components transform as :

$$E'_2 = \gamma(E_2 - vB_3), \quad E'_3 = \gamma(E_3 + vB_2),$$

$$B'_1 = B_1, \quad B'_2 = \gamma(B_2 + vE_3), \quad B'_3 = \gamma(B_3 - vE_2).$$

3. Let in its rest frame the two components of the length vector of the rod along and perpendicular to the direction of motion be  $(l_1, l_2)$ . Then

$$\frac{l_2}{l_1} = \tan 60^\circ = \sqrt{3}.$$

In the frame  $S$ , the component  $l_2$  has the same apparent length  $l'_2$  as before. The length  $l_1$ , however, appears contracted to

$$l'_1 = l_1 \sqrt{1 - \frac{v^2}{c^2}} = l_1 \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4l_1}{5}.$$

Since  $l'_2 = l_2$ , we have the apparent angle of inclination of the rod as  $\theta$ , where

$$\cot \theta = \frac{l'_1}{l'_2} = \frac{4l_1/5}{l_2} = \frac{4}{5} \times \frac{1}{\sqrt{3}},$$

i.e.,  $\theta = \cot^{-1}(4/5\sqrt{3})$ .

4. Let the 4-momentum of the photon in the laboratory frame be

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$$p^i = (E, E \cos \theta, E \sin \theta, 0)$$

After Lorentz transformation in the rest frame of the mirror it becomes

$$p'^i = [\gamma E(1 + v \cos \theta), \gamma E(v + \cos \theta), E \sin \theta, 0].$$

After reflection it will be

$$p'_{\text{ref}}{}^i = [\gamma E(1 + v \cos \theta), -\gamma E(v + \cos \theta), E \sin \theta, 0].$$

Transforming back to the laboratory frame this becomes

$$p_{\text{ref}}{}^i = [\gamma^2 E\{(1 + v \cos \theta) + v(v + \cos \theta)\}, \gamma^2 E\{-v(1 + v \cos \theta) - (v + \cos \theta)\}, E \sin \theta, 0]$$

So  $\cos \bar{\theta}$  after reflection will be

$$\begin{aligned} \cos \bar{\theta} &= \frac{|p^2|}{|p^0|} = \frac{(1 + v^2)\cos \theta + 2v}{1 + 2v \cos \theta + v^2} \\ &= \frac{\cos \theta + \frac{2v}{1 + v^2}}{1 + \frac{2v}{1 + v^2} \cos \theta} \\ &= \frac{\cos \theta + \cos \alpha}{1 + \cos \theta \cos \alpha}. \end{aligned}$$

5. The Compton scattering formula in section 1.7.4 tells us that the wavelength change is given by

$$\Delta\lambda = \frac{h}{m_0 c} (1 - \cos \theta)$$

For  $\theta = 60^\circ$ ,  $\cos \theta = \frac{1}{2}$  and  $\Delta\lambda = h/2m_0 c$ .

6. Going back to the definition of  $F_{ik}$ , we use the fact that the expressions

$$F_{ik} F^{ik} \quad \text{and} \quad \epsilon_{ijkl} F^{ij} F^{kl}$$

are invariants. Substituting the values of the components we find that the first is proportional to  $B^2 - E^2$  and the second to  $\mathbf{B} \cdot \mathbf{E}$ .

7.  $F_{ik}F^{ik}$  and  $\epsilon_{ijkl}F^{ij}F^{kl}$  are invariants and as shown in Q.6, they are  $B^2 - E^2$  and  $\mathbf{B} \cdot \mathbf{E}$ .

Now by a Lorentz transformation we can give arbitrary values to  $B$  and  $E$  subject to the above invariants.

Consider the Lorentz frame in which  $B$  and  $E$  are parallel. Then  $\mathbf{B} \cdot \mathbf{E} = 0$  gives  $BE = 0$ . Hence either  $B = 0$  or  $E = 0$ . That is, either the magnetic or the electric field is zero.

8. The equation of motion of the charge is

$$m \frac{du^i}{ds} = q F^i_k u^k$$

Let the orbit be in  $x^1 - x^2$  plane with the magnetic field in the  $x^3$  - direction.

Then

$$F_{ik} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The 3-velocity of the charge is  $\mathbf{v} = (R\omega \cos \theta, R\omega \sin \theta, 0)$ . This corresponds to a 4-velocity

$$u^i = \frac{1}{\sqrt{1 - R^2\omega^2}}(1, R\omega \cos \theta, R\omega \sin \theta, 0)$$

The equation of motion then has these components:

$$\frac{m}{\sqrt{1 - R^2\omega^2}} \frac{d}{dt} \left( \frac{R\omega \cos \theta}{\sqrt{1 - R^2\omega^2}} \right) = -qB \frac{R\omega \sin \theta}{\sqrt{1 - R^2\omega^2}}$$

$$\frac{m}{\sqrt{1 - R^2\omega^2}} \frac{d}{dt} \left( \frac{R\omega \sin \theta}{\sqrt{1 - R^2\omega^2}} \right) = qB \frac{R\omega \cos \theta}{\sqrt{1 - R^2\omega^2}}$$

Both these equations lead to

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$$\frac{mR\omega}{1 - R^2\omega^2}\dot{\theta} = qB\frac{R\omega}{\sqrt{1 - R^2\omega^2}},$$

Since  $\dot{\theta} = \omega$ ,

$$B = \frac{m\omega}{q\sqrt{1 - R^2\omega^2}}.$$

9. The Doppler spectral shift is given by

$$1 + z = \frac{1 + v \cos \theta}{\sqrt{1 - v^2}}.$$

For zero shift  $z = 0$  and

$$\sqrt{1 - v^2} = 1 + v \cos \theta,$$

$$\text{i.e., } \theta = \cos^{-1} \frac{\sqrt{1 - v^2} - 1}{v}.$$

10. We need to estimate  $v$ , the orbital velocity of the Earth. The Earth–Sun distance is  $\approx 150$  million km  $\approx 1.5 \times 10^{13}$  cm. The circumference of the orbit is (assuming circular shape)

$$l = 2\pi \times 1.5 \times 10^{13} \text{ cm.}$$

This distance is covered by the Earth in time  $T = 1$  year.

$$T \cong 365 \times 24 \times 3600 \text{ second} \approx 3.15 \times 10^7 \text{ s}$$

Therefore

$$v = \frac{2\pi \times 1.5 \times 10^{13}}{3.15 \times 10^7} \cong 3 \times 10^6 \text{ cm s}^{-1}$$

The order of magnitude of the effect, assuming  $\sin \theta = 0(1)$ , is

$$\alpha \cong \frac{2v}{c} \approx \frac{6 \times 10^6}{3 \times 10^{10}} \text{ radians} = 2 \times 10^{-4} \text{ radians} \approx 40 \text{ arsec.}$$

11. The answer is “no”. For, if it were “yes”, in the rest frame of the electron its 4-momentum vector would be  $(m_0c^2, 0, 0, 0)$ . This must equal the sum of 4-momenta of decay products. If the energy of the emitted photon were  $h\nu$  and the electron, after decay, had a  $\gamma$ -factor  $\geq 1$ , we would have

$$m_0c^2 = \gamma m_0c^2 + h\nu.$$

Since the right hand side exceeds the left hand side for  $\gamma \geq 1$ ,  $\nu \geq 0$ , we get a contradiction.

12. The 4-momentum of the original particle is  $(M_0, 0, 0, 0)$ . The decay products have  $\gamma$ -factors,  $\gamma_1, \gamma_2, \gamma_3$  related by

$$\gamma_2 M_2 = \gamma_3 M_3 = X \text{ (say).}$$

Choose the  $x$ -axis along the direction of  $M_2$  and  $y$ -axis along the direction of  $M_3$ . Then the 4-momenta of  $M_2, M_3$  are  $(\gamma_2 M_2, \gamma_2 v_2 M_2, 0, 0)$  and  $(\gamma_3 M_3, 0, \gamma_3 v_3 M_3, 0)$ , respectively. The third decay product will move at an angle  $\pi + \alpha$  with the  $x$ -axis, say, with velocity  $v_1$ . Then its 4-momentum is  $(\gamma_1 M_1, -\gamma_1 M_1 v_1 \cos \alpha, -\gamma_1 M_1 v_1 \sin \alpha, 0)$ . The conservation of momentum then gives these three relations:

$$M_0 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = \gamma_1 M_1 + 2X \text{ where } X = \text{energy of } M_2 \text{ and } M_3.$$

$$0 = -\gamma_1 M_1 v_1 \cos \alpha + \gamma_2 M_2 v_2$$

$$0 = -\gamma_1 M_1 v_1 \sin \alpha + \gamma_3 M_3 v_3.$$

We also have the identity  $\gamma^2 - \gamma^2 v^2 = 1$  for all  $\gamma$ s.

Thus  $\gamma_2^2 M_2^2 - \gamma_2^2 M_2^2 v_2^2 = M_2^2 \Rightarrow X^2 = M_2^2 + X^2 v_2^2$ .

From the second and third momentum equations we get

$$\gamma_1^2 M_1^2 v_1^2 = \gamma_2^2 M_2^2 v_2^2 + \gamma_3^2 M_3^2 v_3^2 = 2X^2 - (M_2^2 + M_3^2),$$

$$\text{i.e., } M_1^2(\gamma_1^2 - 1) = 2X^2 - (M_2^2 + M_3^2),$$