Chapter One

Problem 1.1 (i): Expanding the expression

$$F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$$

Of the three terms on the right hand side, only one is nonzero. It is equal to F_{i1} if $k = 1$, F_{i2} if $k = 2$, or F_{i3} if $k = 3$. Thus, it is simply equal to F_{ik}.

Problem 1.1 (ii):

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Problem 1.1 (iii): By Part (i) of this problem, with $F_{ij} = \delta_{ij}$, the result follows immediately.

Problem 1.1 (iv): Using the $\varepsilon - \delta$ identity, we obtain $(\delta_{ij}\delta_{ij} = \delta_{ii} = 3)$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ij} = 9 - 3 = 6$$

Problem 1.1 (v):

$$A_iA_j\varepsilon_{ijk} = -A_iA_j\varepsilon_{jik} = -A_jA_i\varepsilon_{jik} = -A_iA_j\varepsilon_{ijk}$$

where, in going from the third to the fourth expression, subscript i is renamed as j and j is renamed as i. Since the only real number that is equal to its own negative is zero, the result follows.

Problem 1.1 (vi): The result follows by definition. A cyclic permutation of i, j, and k does not change the value. Interchanging of any two subscripts, which is equivalent to permutation in the opposite direction, changes the sign.

Problem 1.2 (i): We have

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [(A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j)] \times [(C_m\hat{\mathbf{e}}_m) \times (D_n\hat{\mathbf{e}}_n)] \\ &= (A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k) \times (C_mD_n\varepsilon_{mnp}\hat{\mathbf{e}}_p) \\ &= A_iB_jC_mD_n\varepsilon_{ijk}\varepsilon_{mnp}\varepsilon_{kpr}\hat{\mathbf{e}}_r \\ &= A_iB_jC_mD_n\varepsilon_{mnp}\left(\delta_{ip}\delta_{jr} - \delta_{ir}\delta_{jp}\right)\hat{\mathbf{e}}_r \\ &= A_pB_rC_mD_n\varepsilon_{mnp}\hat{\mathbf{e}}_r - A_rB_pC_mD_n\varepsilon_{mnp}\hat{\mathbf{e}}_r\end{aligned}$$

where we have used the $\varepsilon - \delta$ identity. Since $B_r \hat{\mathbf{e}}_r = \mathbf{B}$, $C_m D_n \varepsilon_{mnp} A_p = \mathbf{C} \times \mathbf{D} \cdot \mathbf{A}$, and so on, we can write

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D})]\mathbf{A}$$

Although the above vector identity is established using an orthonormal basis, it holds in a general coordinate system. Also, it can be shown that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{D} \cdot (\mathbf{A} \times \mathbf{B})]\mathbf{C} - [\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})]\mathbf{D}$$

Problem 1.2 (ii): We begin with

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j) \cdot (C_m \hat{\mathbf{e}}_m) \times (D_n \hat{\mathbf{e}}_n) \\
&= (A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k) \cdot (C_m D_n \varepsilon_{mnp} \hat{\mathbf{e}}_p) \\
&= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnp} \delta_{kp} \\
&= A_i B_j C_m D_n \varepsilon_{ijk} \varepsilon_{mnk} \\
&= A_i B_j C_m D_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\
&= A_i B_j C_m D_n \delta_{im}\delta_{jn} - A_i B_j C_m D_n \delta_{in}\delta_{jm}
\end{aligned}$$

where we have used the $\varepsilon - \delta$ identity. Since $C_m \delta_{im} = C_i$ (or $A_i \delta_{im} = A_m$), and $A_i C_i = \mathbf{A} \cdot \mathbf{C}$, and so on, we can write

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_i B_j C_i D_j - A_i B_j C_j D_i \\
&= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})
\end{aligned}$$

New Problem 1.1 ♣: ______

Problem: Show that the dot and cross can be interchanged without changing the value in the *scalar triple product*

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$$

Solution: We have

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} &= A_i \hat{\mathbf{e}}_i \cdot B_j C_k \varepsilon_{jkm} \hat{\mathbf{e}}_m = A_i B_j C_k \varepsilon_{jkm} \delta_{im} \\
&= A_i B_j C_k \varepsilon_{jki} = A_i B_j C_k \varepsilon_{ijk} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}
\end{aligned}$$

Since i, j, and k can be permuted in a cyclic order, it also follows that

$$A_i B_j C_k \varepsilon_{ijk} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A}$$

Problem 1.2 (iii): Using Part (i) of this problem, we can write

$$(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A}) = [\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{C} - [\mathbf{C}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{B} = [\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{C} - 0$$

and then

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot[(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{B})] &= (\mathbf{A}\times\mathbf{B})\cdot[\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\mathbf{C}\\ &= [\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}][\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})]\\ &= [\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}]^2\end{aligned}$$

where the identities of the New Problem 1.1 are used to arrive at the last step.

Problem 1.2 (iv):

$$\begin{aligned}(\mathbf{AB})^{\mathrm{T}} &= (A_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j B_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n)^{\mathrm{T}}\\ &= A_{ij}B_{mn}(\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n)^{\mathrm{T}}\\ &= A_{ij}B_{mn}\hat{\mathbf{e}}_n\hat{\mathbf{e}}_m\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i\\ &= (B_{mn}\hat{\mathbf{e}}_n\hat{\mathbf{e}}_m)(A_{ij}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i)\\ &= (\mathbf{B})^{\mathrm{T}}(\mathbf{A})^{T}\end{aligned}$$

Problem 1.3: Let the new coordinate system be the barred coordinate system. We have

$$\hat{\bar{\mathbf{e}}}_1\cdot\hat{\mathbf{e}}_1 = \cos 90 = a_{11},\quad \hat{\bar{\mathbf{e}}}_1\cdot\hat{\mathbf{e}}_2 = \cos 0 = a_{12},\quad \hat{\bar{\mathbf{e}}}_1\cdot\hat{\mathbf{e}}_3 = \cos 90 = a_{13}$$

$$\hat{\bar{\mathbf{e}}}_2\cdot\hat{\mathbf{e}}_1 = -\cos 45 = a_{21},\quad \hat{\bar{\mathbf{e}}}_2\cdot\hat{\mathbf{e}}_2 = \cos 90 = a_{22},\quad \hat{\bar{\mathbf{e}}}_2\cdot\hat{\mathbf{e}}_3 = \cos 45 = a_{23}$$

$$\hat{\bar{\mathbf{e}}}_3\cdot\hat{\mathbf{e}}_1 = \cos 45 = a_{31},\quad \hat{\bar{\mathbf{e}}}_3\cdot\hat{\mathbf{e}}_2 = \cos 90 = a_{32},\quad \hat{\bar{\mathbf{e}}}_3\cdot\hat{\mathbf{e}}_3 = \cos 45 = a_{33}$$

Thus the transformation matrix is

$$[A] = \begin{bmatrix} 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

Problem 1.4: Using the result of Problem 1.2(iv) and $[A]^T = [A]$, we have

$$([B]^T[A][B])^T = ([A][B])^T([B]^T)^T = [B]^T[A]^T[B] = [B]^T[A][B]$$

Problem 1.5 (i): Since

$$|[A][B]| = |[A]||[B]|$$

and $|[A]| \neq 0$ and $|[B]| \neq 0$, it follows that $|[A][B]| \neq 0$. Hence $[A][B]$ is nonsingular.

Problem 1.5 (ii): By definiton of an inverse we have

$$([A][B])([A][B])^{-1} = [I]$$

Now premultiplying both sides with the inverse of $[A]$, we obtain

$$[A]^{-1}[A][B]([A][B])^{-1} = [A]^{-1}[I] = [A]^{-1}$$

$$[B]([A][B])^{-1} = [A]^{-1}[I] = [A]^{-1}$$

Next premultiply both sides with the inverse of $[B]$ and obtain

$$[B]^{-1}[B]([A][B])^{-1} = [B]^{-1}[A]^{-1} \quad \text{or} \quad ([A][B])^{-1} = [B]^{-1}[A]^{-1}$$

Problem 1.6 (a): The matrix form of the equations is

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -8 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix}$$

Using Cramer's rule we obtain

$$x_1 = \frac{1}{|A|} \begin{vmatrix} 2 & -1 & -1 \\ 2 & 2 & 1 \\ 2 & -8 & -5 \end{vmatrix} = \frac{1}{|A|}[2(-10+8) + (-10-2) - (-16-4)] = \frac{4}{|A|}$$

$$x_2 = \frac{1}{|A|} \begin{vmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 4 & 2 & -5 \end{vmatrix} = \frac{1}{|A|}[2(-10-2) - 2(-5-4) - (2-8)] = \frac{0}{|A|}$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & 2 \\ 4 & -8 & 2 \end{vmatrix} = \frac{1}{|A|}[2(4+16) + (2-8) + 2(-8-8)] = \frac{2}{|A|}$$

where the determinant $|A|$ of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -8 & -5 \end{vmatrix} = 2(-10+8) + (-5-4) - (-8-8) = 3$$

Hence, $x_1 = 4/3, x_2 = 0$, and $x_3 = 2/3$.

Problem 1.6 (b): The matrix form of the equations is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

Using Cramer's rule we obtain

$$x_1 = \frac{1}{|A|}\begin{vmatrix} 1 & -1 & 0 \\ 2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \frac{1}{|A|}[(8-1)+(4+1)-0] = \frac{12}{|A|}$$

$$x_2 = \frac{1}{|A|}\begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{|A|}[2(4+1)-(-2-0)-0] = \frac{12}{|A|}$$

$$x_3 = \frac{1}{|A|}\begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{|A|}[2(4+2)+(-1-0)+(1-0)] = \frac{12}{|A|}$$

where the determinant $|A|$ of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(8-1)+(-2-0)-0 = 12$$

Hence, $x_1 = x_2 = x_3 = 1$.

Problem 1.7: First note the identity

$$\frac{\partial}{\partial x_j}\left[w_i\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] = w_i\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{\partial w_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

Therefore

$$\begin{aligned}
-\int_\Omega w_i\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)dv &= \int_\Omega \frac{\partial w_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)dv - \int_\Omega \frac{\partial}{\partial x_j}\left[w_i\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]dv \\
&= \int_\Omega \frac{\partial w_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)dv - \oint_\Gamma w_i n_j\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)ds
\end{aligned}$$

where Eq. (1.5–2b) is used.

Problem 1.8: Let $\nabla^2\psi = u$. Then using Eq. (1.5–9) [with $\psi = \varphi$, $f = 1$, and $\varphi = u$] we obtain

$$\int_\Omega \varphi\nabla^4\psi\, dv = \int_\Omega \varphi\nabla^2 u\, dv = -\int_\Omega \nabla\varphi\cdot\nabla u\, dv + \oint_\Gamma \varphi\frac{\partial u}{\partial n}\, ds$$

Once again using Eq. (1.5–9)

$$-\int_\Omega \nabla\varphi\cdot\nabla u\, dv = \int_\Omega u\nabla^2\varphi\, dv - \oint_\Gamma u\frac{\partial\varphi}{\partial n}\, ds$$

Thus we have

$$\int_\Omega \varphi\nabla^4\psi\, dv = \int_\Omega \nabla^2\psi\nabla^2\varphi\, dv + \oint_\Gamma \varphi\frac{\partial(\nabla^2\psi)}{\partial n}\, ds - \oint_\Gamma \nabla^2\psi\frac{\partial\varphi}{\partial n}\, ds$$

Problem 1.9 (i): We have

$$
\begin{aligned}
(\mathbf{I} \times \mathbf{A}) \cdot \boldsymbol{\Phi} &= (\delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \times A_k\hat{\mathbf{e}}_k) \cdot (\Phi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n) \\
&= (A_k\varepsilon_{jk\ell}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_\ell) \cdot (\Phi_{mn}\hat{\mathbf{e}}_m\hat{\mathbf{e}}_n) \\
&= A_k\Phi_{mn}\varepsilon_{jk\ell}\delta_{\ell m}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_n \\
&= A_k\Phi_{mn}\varepsilon_{jkm}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_n \\
&= \mathbf{A} \times \boldsymbol{\Phi}
\end{aligned}
$$

Problem 1.9 (ii): We have

$$
\begin{aligned}
(\boldsymbol{\Phi} \times \mathbf{A})^T &= (\Phi_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \times A_k\hat{\mathbf{e}}_k)^T \\
&= (\Phi_{ij}A_k\varepsilon_{jk\ell}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_\ell)^T \\
&= \Phi_{ij}A_k\varepsilon_{jk\ell}\hat{\mathbf{e}}_\ell\hat{\mathbf{e}}_i \\
&= -\,\mathbf{A} \times \boldsymbol{\Phi}^T
\end{aligned}
$$

Problem 1.10: Suppose that $\{X\} \neq \{0\}$ and the determinant of $[A - \lambda I]$ is not zero. Since the determinant is not zero it has an inverse. Therefore we have

$$\{X\} = [A - \lambda I]^{-1}\{0\} = \{0\}$$

which contradicts the hypothesis that $\{X\} \neq \{0\}$.