4.1 A meteoroid descending into the atmosphere travels a distance x before burning up entirely. This distance is a function of (among other things) the meteoroid's initial speed v and its mass m.

In each of the parts below, explain the meaning of the given partial derivative in terms a 12 year old could understand. Also give the units of the derivative and whether you would expect it to be positive or negative.

(a) $\partial x/\partial v$

Solution: This means "How much further will an initially-fast meteoroid get into the atmosphere than an initially-slow meteoroid?" Its units are m/(m/s), which simplifies to s. We would expect it to be positive: if two meteoroids have the same amount of mass to dissipate, then the one that is initially going faster should go further.

Final Answer: positive

(b) $\partial x/\partial m$

Solution: This means "How much further will a heavy meteoroid get into the atmosphere than a light meteoroid?" Its units are m/kg. We would expect it to be positive; a 20-lb boulder should take longer to burn up than a 1-gram pebble, so, if they start at the same speed, the boulder should go further.

Final Answer: positive

(c) List one other factor that the meteoroid's descent distance depends on. Then answer the same questions for the partial derivative with respect to that variable.

Solution: One possible answer would be θ, the angle at which the meteoroid strikes the atmosphere. If θ is defined to be zero for a meteoroid heading straight down then $\partial x/\partial \theta$ means "How much further will a meteoroid that comes in at a shallow angle get than one that is heading more downward?" Its units are m/radians or m/degree (but more properly just m, since radians are unitless). We would expect it to be positive, since a meteoroid heading straight down will get into denser, and thus more destructive, atmosphere sooner. This means a straight-down meteor should go a shorter distance than an angled one before burning up.

Other possible variables could be the density of the meteoroid (positive derivative), the density of the atmosphere (negative derivative), or the temperature of the atmosphere (negative derivative).

4.2 The amount you pay in taxes depends on your income I, your total number of dependents d, and the amount you give to charity c.

In Parts a–c, explain the meaning of the given partial derivative in terms a 12 year old could understand. Also give the units of the derivative and whether you would expect it to be positive or negative.

(a) $\partial T/\partial I$

Solution: This means "How much more will I pay in taxes as my salary grows?" Its units are dollars/dollars, and so is unitless. In other words this number will be the same whether you measure money in dollars or cents.

The number should be positive since the more money you make, the more you pay in taxes, given that your charitable giving and number of dependents stay the same.

Final Answer: positive

(b) $\partial T/\partial d$

Solution: This means "How much more will I pay in taxes as my family grows?" That is, how will having a kid change your taxes? Its units are dollars/dependent. It should be negative, since you are allowed to take tax deductions for each family member you support.

Final Answer: negative

(c) $\partial T/\partial c$

Solution: This means "How much more will I pay in taxes if I donate more to charity?" Its units are dollars/dollars, so it is a unitless derivative. It should be negative, since charitable donations are usually excluded from your overall income for tax purposes.

Final Answer: negative

(d) List one other factor that your taxes depend on. Then answer the same questions for the partial derivative with respect to that variable.

Solution: Perhaps the most obvious answer is r, the tax rate. $\partial T/\partial r$ is a unitless positive quantity: "If the tax rate goes up by 1% how much more money will I pay?"

Another possible answer would be h, the value of your house. $\partial T/\partial h$ means "How much more will I pay in taxes if I move to a more expensive house?" Its units are dollars/dollars, and so is unitless. It should be positive, since you have to pay more in property taxes on more expensive houses.

4.3 Your puppy's weight W depends on its caloric intake c and how many hours per week you walk it, h. In Parts a–b, explain the meaning of the given partial derivative in terms a 12 year old could understand. Also give the units of the derivative and whether you would expect it to be positive or negative.

(a) $\partial W/\partial c$

Solution: This means "How much more will my puppy weigh if I feed it more?" Its units are kg/calories. It should be positive, since more food leads to pudgier puppies.

(b) $\partial W/\partial h$

Solution: This means "How much more will my puppy weigh if I walk it more?" Its units are kg/hours. It should be negative, since exercise burns off the puppy fat.

(c) List one other factor that your puppy's weight depends on. Then answer the same questions for the partial derivative with respect to that variable.

Solution: One solution would be the number of intestinal parasites the puppy has. Its units are kg/worm. It should be negative, since parasites absorb nutrients that would otherwise go to puppy.

4.4 Consider a function $u(x, y, z, t)$ that gives the temperature of the air in a room as a function of time.

(a) What would it mean physically if $\partial u/\partial z > 0$ at some point (x_0, y_0, z_0, t_0)?
Solution: At that moment, the temperature slightly above this point is higher than the temperature at this point.

(b) What would it mean physically if $\partial u/\partial t = 0$ at some point (x_0, y_0, z_0, t_0)?
Solution: The temperature at that point is not changing right now.

(c) Is it possible for both of the above statements to be true at the same place and time?
Final Answer: yes

4.5 The temperature u on Skullcrusher Mountain depends on height h and time t.

(a) Everywhere on the mountain $\partial u/\partial h$ is negative. What does that tell you about the mountain?

Solution: Mountains are colder toward the top than at the bottom.

(b) Throughout the morning $\partial u/\partial t$ is positive. What does that tell you about the mountain?

Solution: The air warms all over the mountain as the sun rises.

(c) Suppose $\partial^2 u/\partial h^2$ is negative. What would that mean? (Explain in the clearest and least technical language you can, for the benefit of a mountain-climber who knows no calculus.)

Solution: As you walk up the mountain the temperature keeps dropping faster and faster. So for instance the temperature drops a certain amount as you go from the bottom of the mountain to the middle, but it drops even more as you go from the middle of the mountain to the top.

(d) Suppose $\partial^2 u/\partial t^2$ is negative. What would that mean? (Same comment.)

Solution: As the sun rises the mountain will warm up quickly at first, but then more and more slowly as the day wears on.

(e) Suppose $\partial^2 u/(\partial t \partial h)$ is zero. What would that mean? (Same comment.) *There are two equally valid answers you could give to this question.*

Solution: Answer 1: The sun warms up the mountain equally fast on top and bottom.
Answer 2: The temperature drop as you climb the mountain is the same no matter what time of day you climb it.
It isn't immediately obvious that these two answers are equivalent, is it? Both of them represent the same equation, so they must be. See if you can convince yourself that either of these statements implies the other.

4.6 $f = xy^2$

Solution: To find $\partial f/\partial x$, hold y constant and take the derivative of f with respect to x.

$$\frac{\partial f}{\partial x} = y^2$$

To find $\partial f/\partial y$, hold x constant and take the derivative of f with respect to y.

$$\frac{\partial f}{\partial y} = 2xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = 2y$$

Final Answer: $\partial f/\partial x = y^2$, $\partial f/\partial y = 2xy$, $\partial^2 f/\partial x\partial y = 2y$

4.7 $f = x/y$

Solution: To find $\partial f/\partial x$, hold y constant and take the derivative of f with respect to x.

$$\frac{\partial f}{\partial x} = \frac{1}{y}$$

To find $\partial f/\partial y$, hold x constant and take the derivative of f with respect to y.

$$\frac{\partial f}{\partial y} = \frac{-x}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{-1}{y^2}$$

Final Answer: $\partial f/\partial x = 1/y$, $\partial f/\partial y = -x/y^2$, $\partial^2 f/\partial x\partial y = -1/y^2$

4.8 $f = a\cos(bxy) + ce^{d/y}$

Solution: Remember that a, b, c, and d are constants.

To find $\partial f/\partial x$, hold y constant and take the derivative of f with respect to x.

$$\frac{\partial f}{\partial x} = -aby\sin(bxy)$$

To find $\partial f/\partial y$, hold x constant and take the derivative of f with respect to y.

$$\frac{\partial f}{\partial y} = -abx\sin(bxy) - \frac{cd}{y^2}e^{d/y}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -ab(\sin(bxy) + bxy\cos(bxy))$$

Final Answer: $\partial f/\partial x = -aby\sin(bxy)$, $\partial f/\partial y = -abx\sin(bxy) - [(cd)/y^2]e^{d/y}$, $\partial^2 f/\partial x\partial y = \partial/\partial x(\partial f/\partial y) = -ab(\sin(bxy) + bxy\cos(bxy))$

4.9 $f = \sin x / \cos y$
Solution: To find $\partial f / \partial x$, hold y constant and take the derivative of f with respect to x.

$$\frac{\partial f}{\partial x} = \frac{\cos x}{\cos y}$$

To find $\partial f / \partial y$, hold x constant and take the derivative of f with respect to y.

$$\frac{\partial f}{\partial y} = \frac{\sin x \sin y}{\cos^2 y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\cos x \sin y}{\cos^2 y}$$

Final Answer: $\partial f / \partial x = \cos x / \cos y$, $\partial f / \partial y = (\sin x \sin y)/(\cos^2 y)$, $\partial^2 f / \partial x \partial y = (\cos x \sin y)/(\cos^2 y)$

4.10 If $f(x, y, t) = ae^{bx^2y}\sin(\omega t)$ find $\partial f/\partial t$ and $\partial^2 f/\partial x \partial y$.

Solution: Remember that a, b, and ω are constants.

To find $\partial f/\partial t$, hold x and y constant and take the derivative of f with respect to t.

$$\frac{\partial f}{\partial t} = a\omega e^{bx^2y}\cos(\omega t)$$

To find $\partial f/\partial x$, hold y and t constant and take the derivative of f with respect to x.

$$\frac{\partial f}{\partial x} = 2abxye^{bx^2y}\sin(\omega t)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = 2abxe^{bx^2y}\sin(\omega t) + 2ab^2x^3ye^{bx^2y}\sin(\omega t)$$

Final Answer: $\partial f/\partial t = a\omega e^{bx^2y}\cos(\omega t)$, $\partial^2 f/\partial x \partial y = 2abxe^{bx^2y}\sin(\omega t)\left(1 + bx^2y\right)$