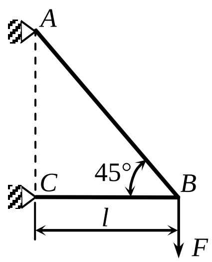
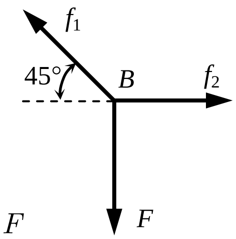
CHAP 1.
STRESS–STRAIN ANALYSIS

1. A vertical force F is applied to a two-bar truss as shown in the figure. Let cross-sectional areas of the members 1 and 2 be A_1 and A_2, respectively. Determine the area ratio A_1/A_2 in order to have the same magnitude of stress in both members.



Solution:

From force equilibrium at B,



$$\sum F_y = f_1 \sin 45^\circ - F = 0 \quad \Rightarrow \quad f_1 = \sqrt{2}F$$

$$\sum F_x = f_2 - f_1 \cos 45^\circ = 0 \quad \Rightarrow \quad f_2 = \sqrt{2}F \cdot \frac{1}{\sqrt{2}} = F$$

Since truss is two-force member, $f_1 = A_1\sigma_1$ and $f_2 = A_2\sigma_2$. Thus,

$$\frac{f_1}{f_2} = \frac{\sqrt{2}F}{F} = \frac{A_1\sigma_1}{A_2\sigma_2} = \frac{A_1}{A_2} (\because \sigma_1 = \sigma_2),$$

$$\therefore \frac{A_1}{A_2} = \sqrt{2}$$

2. The stress at a point P is given below. The direction cosines of the normal **n** to a plane that passes through P have the ratio n_x:n_y:n_z = 3:4:12. Determine (a) the traction vector $\mathbf{T}^{(\mathbf{n})}$; (b) the magnitude T of $\mathbf{T}^{(\mathbf{n})}$; (c) the normal stress σ_n; (d) the shear stress τ_n; and (e) the angle between $\mathbf{T}^{(\mathbf{n})}$ and **n**.

   Hint: Use $n_x^2 + n_y^2 + n_z^2 = 1$.

$$[\sigma] = \begin{bmatrix} 13 & 13 & 0 \\ 13 & 26 & -13 \\ 0 & -13 & -39 \end{bmatrix}$$

Solution:

(a) First, we need unit normal vector **n**:

$$\{\mathbf{n}\} = \frac{1}{\sqrt{3^2 + 4^2 + 12^2}} \begin{Bmatrix} 3 \\ 4 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix}$$

Then, the traction vector on this plane becomes

$$\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}]\{\mathbf{n}\} = \begin{bmatrix} 13 & 13 & 0 \\ 13 & 26 & -13 \\ 0 & -13 & -39 \end{bmatrix} \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix} = \begin{Bmatrix} 7 \\ -1 \\ -40 \end{Bmatrix}$$

(b) Since $\mathbf{T}^{(\mathbf{n})}$ is a vector, its magnitude can be obtained using the norm as

$$\left\| T^{(n)} \right\| = \sqrt{T_x^{(n)2} + T_y^{(n)2} + T_z^{(n)2}} = \sqrt{7^2 + (-1)^2 + (-40)^2} = 40.6202$$

(c)
$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \{\mathbf{n}\} = \{7 \quad -1 \quad -40\} \begin{Bmatrix} 0.2308 \\ 0.3077 \\ 0.9231 \end{Bmatrix} = -35.6154$$

(d)
$$\tau_n = \sqrt{\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 - \sigma_n{}^2} = \sqrt{40.6202^2 - (-35.6154)^2} = 19.5331$$

(e)
$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \{\mathbf{n}\} = \left\| \mathbf{T}^{(\mathbf{n})} \right\| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\sigma_n}{\left\| T^{(n)} \right\|} \right) = 2.64 \approx 151.3^0$$

3. At a point P in a body, Cartesian stress components are given by σ_{xx} = 80 MPa, σ_{yy} = −40 MPa, σ_{zz} = −40 MPa, and $\tau_{xy} = \tau_{yz} = \tau_{zx}$ = 80 MPa. Determine the traction vector, its normal component, and its shear component on a plane that is equally inclined to all three coordinate axes.

   Hint: When a plane is equally inclined to all three coordinate axes, the direction cosines of the normal are equal to each other.

Solution:

The unit normal in this case is:

$$\left\{\mathbf{n}\right\} = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \begin{Bmatrix}0.577\\0.577\\0.577\end{Bmatrix}$$

The traction vector in this direction becomes

$$\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}]\{\mathbf{n}\} = \begin{bmatrix}80 & 80 & 80\\80 & -40 & 80\\80 & 80 & -40\end{bmatrix}\begin{Bmatrix}0.577\\0.577\\0.577\end{Bmatrix} = \begin{Bmatrix}138.56\\69.28\\69.28\end{Bmatrix}\mathrm{MPa}$$

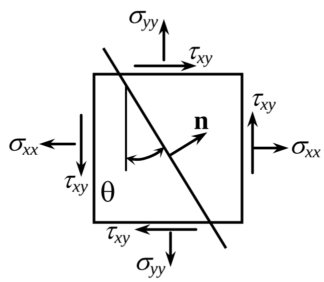
The normal component of the traction vector is

$$\sigma_n = \mathbf{T}^{(\mathbf{n})}\cdot\left\{\mathbf{n}\right\} = \left\{138.56 \quad 69.28 \quad 69.28\right\}\begin{Bmatrix}0.577\\0.577\\0.577\end{Bmatrix} = 160\,\mathrm{MPa}$$

The shear component of the traction vector is:

$$\tau_n = \sqrt{\left\|\mathbf{T}^{(\mathbf{n})}\right\|^2 - {\sigma_n}^2} = \sqrt{169.71^2 - 160^2} = 56.58\ \mathrm{MPa}$$

4. If σ_{xx} = 90 MPa, σ_{yy} = −45 MPa, τ_{xy} = 30 MPa, and $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, compute the surface traction $\mathbf{T}^{(\mathbf{n})}$ on the plane shown in the figure, which makes an angle of θ = 40° with the vertical axis. What are the normal and shear components of stress on this plane?



Solution:

Unit normal vector: $\mathbf{n}^T = \begin{Bmatrix} \cos(40) & \sin(40) & 0 \end{Bmatrix}$

Traction vector: $\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}] \cdot \mathbf{n} = \begin{bmatrix} 90 & 30 & 0 \\ 30 & -45 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} .776 \\ .643 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 88.23 \\ -5.94 \\ 0 \end{Bmatrix} \text{MPa}$

Normal stress: $\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = 63.77\text{MPa}$

Shear stress: $\tau_n = \sqrt{\left|\mathbf{T}^{(\mathbf{n})}\right|^2 - \sigma_n^2} = 61.27\text{MPa}$

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5. Find the principal stresses and the corresponding principal stress directions for the following cases of plane stress.

   (a) $\sigma_{xx} = 40$ MPa, $\sigma_{yy} = 0$ MPa, $\tau_{xy} = 80$ MPa

   (b) $\sigma_{xx} = 140$ MPa, $\sigma_{yy} = 20$ MPa, $\tau_{xy} = -60$ MPa

   (c) $\sigma_{xx} = -120$ MPa, $\sigma_{yy} = 50$ MPa, $\tau_{xy} = 100$ MPa

Solution:

(a) The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 40 & 80 \\ 80 & 0 \end{bmatrix} \mathrm{MPa}$$

To find the principal stresses, the standard eigen value problem can be written as

$$\left[\boldsymbol{\sigma} - \sigma\mathbf{I}\right]\left\{\mathbf{n}\right\} = 0$$

The above problem will have non-trivial solution when the determinant of the coefficient matrix becomes zero:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = \begin{vmatrix} 40 - \sigma & 80 \\ 80 & 0 - \sigma \end{vmatrix} = 0$$

The equation of the determinant becomes:

$$\left(\left(40 - \sigma\right) \cdot -\sigma\right) - \left(80 \cdot 80\right) = \sigma^2 - 40\sigma - 6400 = 0$$

The above quadratic equation yields two principal stresses, as

$$\sigma_1 = 102.46\,\mathrm{MPa} \text{ and } \sigma_2 = -62.46\,\mathrm{MPa}.$$

To determine the orientation of the first principal stresses, substitute σ_1 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 - 102.46 & 80 \\ 80 & 0 - 102.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Since the determinant is zero, two equations are not independent

$$62.46 \cdot n_x = 80 \cdot n_y \text{ and } 80 \cdot n_x = -102.46 \cdot n_y.$$

Thus, we can only get the relation between n_x and n_y. Then using the condition $|\mathbf{n}| = 1$ we obtain

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.788 \\ 0.615 \end{Bmatrix}$$

To determine the orientation of the second principal stress, substitute σ_2 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 + 62.46 & 80 \\ 80 & 0 + 62.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$102.46 \cdot n_x = -80 \cdot n_y \text{ and } 80 \cdot n_x = -62.46 \cdot n_y .$$

Using similar procedures as above, the eigen vector of σ_2 can be obtained as

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.615 \\ -0.788 \end{Bmatrix}$$

Note that if **n** is a principal direction, −**n** is also a principal direction

(b) Repeat the procedure in (a) to obtain

$$\sigma_1 = 164.85\,\text{MPa} \text{ and } \sigma_2 = -4.85\,\text{MPa} .$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} -0.924 \\ 0.383 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.383 \\ 0.924 \end{Bmatrix}$$

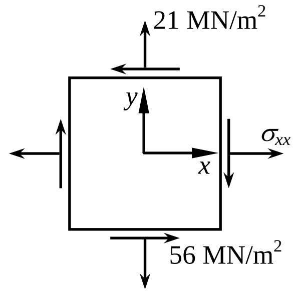
(c) Repeat the procedure in (a) to obtain

$$\sigma_1 = 96.24\,\text{MPa} \text{ and } \sigma_2 = -166.24\,\text{MPa} .$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.420 \\ 0.908 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} -0.908 \\ 0.420 \end{Bmatrix}$$

Note that for the case of plane stress σ_3=0 is also a principal stress and the corresponding principal stress direction is given by $n^{(3)}$ =(0,0,1)

6. If the minimum principal stress is −7 MPa, find σ_{xx} and the angle that the principal stress axes make with the x and y axes for the case of plane stress illustrated



Solution:

With unknown x-component, the eigen value problem can be written as

$$\begin{bmatrix} \sigma_{xx} - \sigma & -56 \\ -56 & 21 - \sigma \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The principal stresses can be determined by making the determinant zero

$$\begin{vmatrix} \sigma_{xx} - \sigma & -56 \\ -56 & 21 - \sigma \end{vmatrix} = 0 \quad \Rightarrow \quad (\sigma_{xx} - \sigma)(21 - \sigma) - 56^2 = 0$$

Since −7 MPa is one of the roots of the above equation, we can find σ_{xx} by substituting in the above equation as

$$(\sigma_{xx} + 7)(21 + 7) - 56^2 = 0$$

By solving the above equation, we can get $\sigma_{xx} = 105\,\text{MPa}$. Then, the other principal stress can be found from the original determinant, as

$$\sigma_1 = 133\,\text{MPa} \qquad \sigma_2 = -7\,\text{MPa}$$

Principal direction for the first principal stress: From the original eigen value problem,

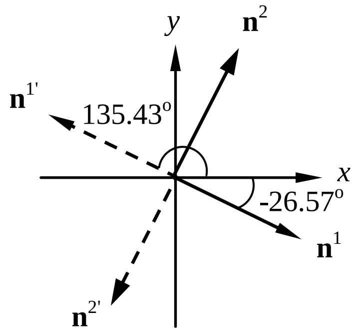
$$\begin{aligned} (105 - 133)n_x^1 - 56n_y^1 &= 0 \\ -56n_x^1 + (21 - 133)n_y^1 &= 0 \end{aligned}$$

The solution of the above equations is not unique. By putting $|\mathbf{n}^1| = 1$, we have $\mathbf{n}^1 = \{\pm 0.8944, \mp 0.4472\}$, which is principal direction corresponding to σ_1

Principal direction for the second principal stress: From the original eigen value problem,

$$(105+7)n_x^2 - 56n_y^2 = 0$$
$$-56n_x^2 + (21+7)n_y^2 = 0$$

The solution of above equations is $\mathbf{n}^2 = \{\pm 0.4472, \pm 0.8944\}$, which is principal direction corresponding to σ_2 . Two principal directions are plotted on the following graph. Note that the two principal directions are perpendicular each other.



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7. Determine the principal stresses and their associated directions, when the stress matrix at a point is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \mathrm{MPa}$$

Solution:

Use the Eq. (0.46) of Chapter 0 with the coefficients of I_1=3, I_2= −3, and I_3 = −1,

$$\lambda^3 - 3\lambda^2 - 3\lambda + 1 = 0$$

By solving the above cubic equation using the method described in Section 0.4,

$$\sigma_1 = 3.73\,\mathrm{MPa}, \quad \sigma_2 = 0.268\,\mathrm{MPa}, \quad \sigma_3 = -1.00\,\mathrm{MPa}$$

(a) Principal direction corresponding to σ_1:

$$\begin{aligned} (1 - 3.7321)n_x^1 + n_y^1 + n_z^1 &= 0 \\ n_x^1 + (1 - 3.7321)n_y^1 + 2n_z^1 &= 0 \\ n_x^1 + 2n_y^1 + (1 - 3.7321)n_z^1 &= 0 \end{aligned}$$

Solving the above equations with $|\mathbf{n}^1| = 1$ yields

$$\mathbf{n^1} = \{\pm 0.4597, \pm 0.6280, \pm 0.6280\}$$

(b) Principal direction corresponding to σ_2:

$$\begin{aligned} (1 - 0.2679)n_x^1 + n_y^1 + n_z^1 &= 0 \\ n_x^2 + (1 - 0.2679)n_y^2 + 2n_z^2 &= 0 \\ n_x^2 + 2n_y^2 + (1 - 0.2679)n_z^2 &= 0 \end{aligned}$$

Solving the above equations with $|\mathbf{n}^2| = 1$ yields

$$\mathbf{n}^2 = \{\pm 0.8881, \mp 0.3251, \mp 0.3251\}$$

(c) Principal direction corresponding to σ_3:

$$\begin{aligned} (1 + 1)n_x^3 + n_y^3 + n_z^3 &= 0 \\ n_x^3 + (1 + 1)n_y^3 + 2n_z^3 &= 0 \\ n_x^3 + 2n_y^3 + (1 + 1)n_z^3 &= 0 \end{aligned}$$

Solving the above equations with $|\mathbf{n}^2| = 1$ yields

$$\mathbf{n}^3 = \{0, \pm 0.7071, \mp 0.7071\}$$

8. Let $x'y'z'$ coordinate system be defined using the three principal directions obtained from Problem 7. Determine the transformed stress matrix $[\boldsymbol{\sigma}]_{x'y'z'}$ in the new coordinates system.

Solution:

The three principal directions in Problem 6 can be used for the coordinate transformation matrix:

$$\left[\mathbf{N}\right]=\begin{bmatrix} n_x^{(1)} & n_x^{(2)} & n_x^{(3)} \\ n_y^{(1)} & n_y^{(2)} & n_y^{(3)} \\ n_z^{(1)} & n_z^{(2)} & n_z^{(3)} \end{bmatrix}=\begin{bmatrix} 0.460 & -0.888 & 0 \\ 0.628 & 0.325 & 0.707 \\ 0.628 & 0.325 & -0.707 \end{bmatrix}$$

To determine the stress components in the new coordinates we use Eq. (1.30):

$$\left[\boldsymbol{\sigma}\right]_{x'y'z'}=\left[\mathbf{N}\right]^T\left[\boldsymbol{\sigma}\right]\left[\mathbf{N}\right]=\begin{bmatrix} -1 & 0 & 0 \\ 0 & .268 & 0 \\ 0 & 0 & 3.732 \end{bmatrix}$$

Note that the transformed stress matrix is a diagonal matrix with the original principal stresses on the diagonal.

9. For the stress matrix below, the two principal stresses are given as $\sigma_3 = -3$ and $\sigma_1 = 2$, respectively. In addition, the two principal stress directions corresponding to the two principal stresses are also given below.

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}, \quad \mathbf{n}^1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \text{ and } \mathbf{n}^3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$$

(a) What is the normal and shear stress on a plane whose normal vector is parallel to (2, 1, 2)?

(b) Calculate the missing principal stress σ_2 and the principal direction $\mathbf{n}^2$.

(c) Write stress matrix in the new coordinates system that is aligned with $\mathbf{n}^1$, $\mathbf{n}^2$, and $\mathbf{n}^3$.

Solution:

(a) Normal vector: $\mathbf{n}^T = \frac{1}{3}\{2 \quad 1 \quad 2\}$

Traction vector $\mathbf{T}^{(\mathbf{n})} = [\boldsymbol{\sigma}]\mathbf{n} = \begin{Bmatrix} 2 \\ \frac{1}{3} \\ 0 \end{Bmatrix}$

The normal component of the stress vector on the plane can be calculated

$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = 1.4444$$
$$\tau_n = \sqrt{\left|\mathbf{T}^{(\mathbf{n})}\right|^2 - \sigma_n^2} = 1.4229$$

(b) Using Eq. (0.46) of Chapter 0, the eigen values are governed by

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

We can find the coefficients of the above cubic equation from Eq. (0.47) by $I_1 = 0$, $I_2 = -7$, and $I_3 = -6$. Thus, we have

$$\lambda^3 - 7\lambda + 6 = (\lambda - 1)(\lambda^2 + \lambda - 6) = 0$$

Thus, the missing principal stress is $\sigma_2 = 1$.

Since three principal directions are mutually orthogonal, the third principal direction can be calculated using the cross product. To establish a defined sign convention for the principal axes, we require them to form a right-handed triad. If $\mathbf{n}^1$ and $\mathbf{n}^3$ are unit vectors that define the directions of the first and third principal axes, then the unit vector $\mathbf{n}^2$ for the second principal axis is determined by the right-hand rule of the vector multiplication. Thus we have

$$\mathbf{n}^2 = \mathbf{n}^3 \times \mathbf{n}^1 = \{0 \quad 1 \quad 0\}^T$$

(c) Coordinate transformation matrix can be obtained from three principal directions as

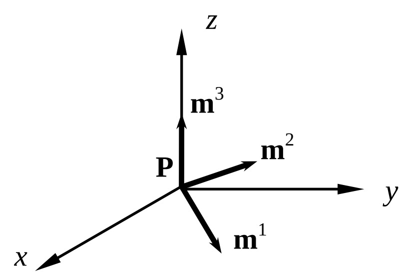
$$\left[\mathbf{N}\right] = \left[\mathbf{n^1} \quad \mathbf{n^2} \quad \mathbf{n^3}\right] = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

The stress matrix at the transformed coordinates becomes

$$\left[\mathbf{N}\right]^T \left[\boldsymbol{\sigma}\right]\left[\mathbf{N}\right] = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

10. With respect to the coordinate system *xyz*, the state of stress at a point *P* in a solid is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} -20 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{MPa}$$



(a) $\mathbf{m}^1$, $\mathbf{m}^2$ and $\mathbf{m}^3$ are three mutually perpendicular vectors such that $\mathbf{m}^1$ makes 45° with both *x*- and *y*-axes and $\mathbf{m}^3$ is aligned with the *z*-axis. Compute the normal stresses on planes normal to $\mathbf{m}^1$, $\mathbf{m}^2$, and $\mathbf{m}^3$.

(b) Compute two components of shear stress on the plane normal to $\mathbf{m}^1$ in the directions $\mathbf{m}^2$ and $\mathbf{m}^3$.

(c) Is the vector $\mathbf{n} = \{0, 1, 1\}^T$ a principal direction of stress? Explain. What is the normal stress in the direction $\mathbf{n}$?

(d) Draw an infinitesimal cube with faces normal to $\mathbf{m}^1$, $\mathbf{m}^2$ and $\mathbf{m}^3$ and display the stresses on the positive faces of this cube.

(e) Express the state of stress at the point *P* with respect to the *x′y′z′* coordinates system that is aligned with the vectors $\mathbf{m}^1$, $\mathbf{m}^2$ and $\mathbf{m}^3$?

(f) What are the principal stress and principal directions of stress at the point *P* with respect to the *x′y′z′* coordinates system? Explain.

(g) Compute the maximum shear stress at the point *P*. Which plane(s) does this maximum shear stress act on?

Solution:

(a)

$$\mathbf{m}^1 = \frac{1}{\sqrt{2}}(1,1,0)^T \qquad \mathbf{m}^2 = \frac{1}{\sqrt{2}}(-1,1,0)^T \qquad \mathbf{m}^3 = (0,0,1)^T$$

$$\begin{aligned} \sigma_{\mathbf{m}^1\mathbf{m}^1} &= \mathbf{m}^1 \cdot [\sigma] \cdot \mathbf{m}^1 = 15 \text{ MPa} \\ \sigma_{\mathbf{m}^2\mathbf{m}^2} &= \mathbf{m}^2 \cdot [\sigma] \cdot \mathbf{m}^2 = 15 \text{ MPa} \\ \sigma_{\mathbf{m}^3\mathbf{m}^3} &= \mathbf{m}^3 \cdot [\sigma] \cdot \mathbf{m}^3 = 50 \text{ MPa} \end{aligned}$$

(b)

$$\mathbf{T}^{(\mathbf{m}^1)} = [\boldsymbol{\sigma}] \cdot \mathbf{m}^1 = \frac{1}{\sqrt{2}}\{-20 \quad 50 \quad 0\}^T$$

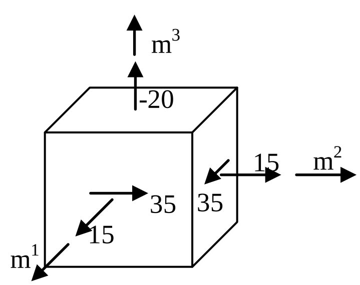
$$\tau_{\mathbf{m}^1\mathbf{m}^2} = \mathbf{T}^{(\mathbf{m}^1)} \cdot \mathbf{m}^2 = 35 \text{ MPa}$$
$$\tau_{\mathbf{m}^1\mathbf{m}^3} = \mathbf{T}^{(\mathbf{m}^1)} \cdot \mathbf{m}^3 = 0 \text{ MPa}$$

(c) Yes,

$$\mathbf{n} = \frac{1}{\sqrt{2}}(0,1,1)$$

$$\mathbf{T}^{(\mathbf{n})} = [\sigma]\cdot\mathbf{n} = \frac{1}{\sqrt{2}}\begin{Bmatrix}50\\50\\0\end{Bmatrix} = 50\times\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\1\\0\end{Bmatrix} = 50\mathbf{n}$$

Since $\mathbf{T}^{(\mathbf{n})}$ // **n**, **n** is a principal direction with principal stress = 50 MPa.
(d)



(e)

$$[\sigma]_{x'y'z'} = [N]^T[\sigma][N] = \begin{bmatrix}0.707 & 0.707 & 0\\ -0.707 & 0.707 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-20 & 0 & 0\\ 0 & 50 & 0\\ 0 & 0 & 50\end{bmatrix} \times\begin{bmatrix}0.707 & -0.707 & 0\\ 0.707 & 0.707 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\sigma]_{x'y'z'} = \begin{bmatrix}15 & 35 & 0\\ 35 & 15 & 0\\ 0 & 0 & 50\end{bmatrix}\text{MPa}$$

(f) Principal stresses = 50, 50, and −20 MPa

$$\mathbf{n}^3 = \frac{1}{\sqrt{2}}(1,-1,0)$$

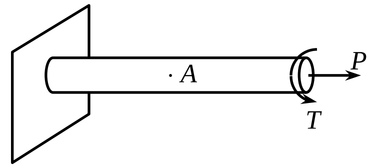
$\mathbf{n}^1$ and $\mathbf{n}^2$ are any two perpendicular unit vectors that is on the plane perpendicular to $\mathbf{n}^3$.

(g) The maximum shear stress occurs on a plane whose normal is at 45° from the principal stress direction. Since $\sigma_1 = \sigma_2$, all directions that are 45° from *x*-axis (σ_3 axis) will have the maximum shear stress whose value is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 35\,\mathrm{MPa}$$

The maximum shear stress planes are in the shape of a cone whose axis is parallel to *x*-axis and has an angle of 45°.

11. A solid shaft of diameter d = 5 cm, as shown in the figure, is subjected to tensile force P = 13,000 N and a torque T = 6,000 N·cm. At point A on the surface, what is the state of stress (write in matrix form), the principal stresses, and the maximum shear stress? Show the coordinate system you are using.

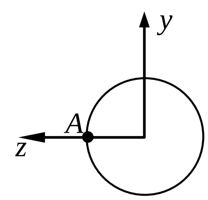


Solution:

Let us establish a coordinate system as shown in the figure. The axial force will cause normal stress σ_{xx}, while the torque will cause shear stress τ_{xy}. Their magnitudes are

$$\frac{P}{A} = 6.62 \text{ MPa}$$

$$\frac{T \cdot r}{J} = 2.44 \text{ MPa}$$



Then, the stress matrix becomes

$$[\sigma]_A = \begin{bmatrix} 6.62 & -2.44 & 0 \\ -2.44 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

By solving the eigen value problem, the principal stress can be obtained as

$$\sigma_1 = 7.43, \quad \sigma_2 = 0, \quad \sigma_3 = -0.81 \text{ MPa}$$

Maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 4.11 \text{ MPa}$$

12. If the displacement field is given by

$$\begin{cases} u_x = x^2 + 2y^2 \\ u_y = -y^2 - 2x(y-z) \\ u_z = -z^2 - 2xy \end{cases}$$

(a) Write down 3×3 strain matrix.

(b) What is the normal strain component in the direction of (1,1,1) at point (1,–3,1)?

Solution:

(a) 3×3 symmetric strain matrix can be calculated from its definition as

$$[\varepsilon] = \begin{bmatrix} 2x & y+z & -y \\ y+z & -2(x+y) & 0 \\ -y & 0 & -2z \end{bmatrix}$$

In addition, the unit normal vector in the direction of (1, 1, 1) is

$$\mathbf{n}^T = \frac{1}{\sqrt{3}}\{1 \quad 1 \quad 1\}$$

b) Thus, the normal component of strain is

$$\mathbf{n} \cdot [\varepsilon] \cdot \mathbf{n} = \frac{1}{3}(2x + y + z - y + y + z - 2x - 2y - y - 2z) = -\frac{2}{3}y$$

Thus, the normal component of strain reduces as the *y*-coordinate of a point increases. At point (1, −3, 1), $y = -3$

$$\therefore \quad \mathbf{n} \cdot [\varepsilon] \cdot \mathbf{n}\big|_{y=-3} = 2\,.$$

13. Consider the following displacement field in a plane solid:

$$u(x,y) = 0.04 - 0.01x + 0.006y$$
$$v(x,y) = 0.06 + 0.009x + 0.012y$$

(a) Compute the strain components ε_{xx}, ε_{yy}, and γ_{xy}. Is this a state of uniform strain?

(b) Determine the principal strains and their corresponding directions. Express the principal strain directions in terms of angles the directions make with the *x*-axis.

(c) What is the normal strain at Point *O* in a direction 45° to the *x*-axis?

Solution:

(a) Strain components:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -0.01$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0.012$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.009 + 0.006 = 0.015$$

Yes, this is a state of uniform strain, because the strains are independent of position *x*,*y*,*z*.

(b) Principal strains and principal directions.

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = 0.0075$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix}$$

Find the eigen values (principal strains) and eigen vectors (principal direction) by solving the eigen value problem:

$$\begin{bmatrix} -0.01-\lambda & 0.0075 \\ 0.0075 & 0.012-\lambda \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The above equation yields two principal strains: $\varepsilon_1 = \lambda_1 = -0.01231$ and $\varepsilon_2 = \lambda_2 = 0.01431$. The principal direction corresponding to the first principal strain is

$$\mathbf{n}^{(1)} = \begin{Bmatrix} -0.9556 & 0.2948 \end{Bmatrix},$$

The angle the direction makes with the *x*-axis can be found from the relation $\cos\theta = -0.9556, \sin\theta = +0.2948$. Solving $\theta \approx 163^o$

The principal direction corresponding to the second principal strain is

$$\mathbf{n}^{(2)} = \left\{ 0.2948 \quad 0.9556 \right\},$$

and the angle is found to be $\theta \approx 73^o$

(c)

Strain at point O
$$\boldsymbol{\varepsilon} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix},$$

direction vector
$$\mathbf{n} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}$$

Thus the normal strain in the direction of **n** becomes

$$\varepsilon_{45^o} = \mathbf{n} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{n} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}^T \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix} \cdot \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix} = 0.0085$$