CHAPTER 2 (odd-numbered problems)

Instructor's Solutions Manual

to accompany

Fundamentals of Structural Dynamics

Second Edition

by

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Prepared by the authors

2.1 Solution

(a) Determine the equation of motion of the mass m in Fig. 1.

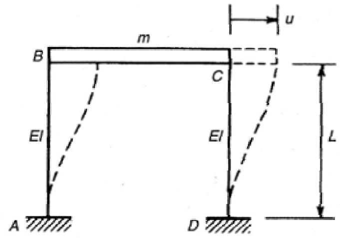

Figure 1

To determine the stiffness of the columns, we can use the unit dummy load method.

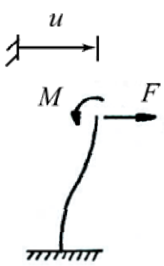

Figure 2

$$u = \int_0^L \frac{(-Fx)(-x)}{EI}\,dx + \int_0^L \frac{M(-x)}{EI}\,dx$$
$$= \frac{FL^3}{3EI} - \frac{ML^2}{2EI}$$
$$\theta = \int_0^L \frac{(-Fx)(1)}{EI}\,dx + \int_0^L \frac{M(1)}{EI}\,dx$$
$$= -\frac{FL^2}{2EI} + \frac{ML}{EI}$$

Since $\theta = 0$,

$$M = \frac{FL}{2} \quad \text{and} \quad F = \frac{12EI}{L^3}u = ku$$

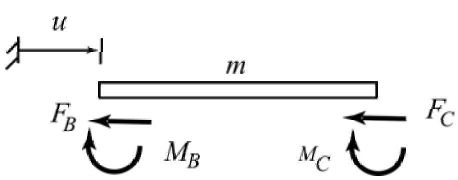

Figure 3

$$\sum F_x = m\ddot{u}$$

From the free-body diagram in Fig. 3,

$$-F_B - F_C = -\frac{12EI}{L^3}u - \frac{12EI}{L^3}u = m\ddot{u}$$

$$m\ddot{u} + \frac{24EI}{L^3}u = 0$$ **Ans. (a)**

(b) How would the equation of motion differ if the left-hand column had a stiffness $2EI$, with no other changes to the structure?

$$-F_B - F_C = -\frac{12(2EI)}{L^3}u - \frac{12EI}{L^3}u = m\ddot{u}$$

$$m\ddot{u} + \frac{36EI}{L^3}u = 0$$ **Ans. (b)**

2.3 Solution

(a) Determine the equation of motion of the beam in Fig. 1 in terms of tip displacement v.

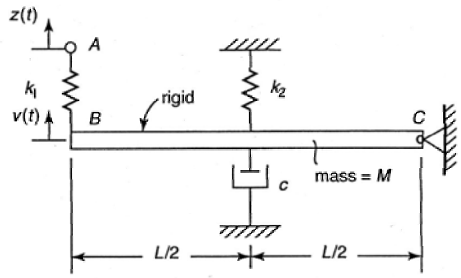

Figure 1

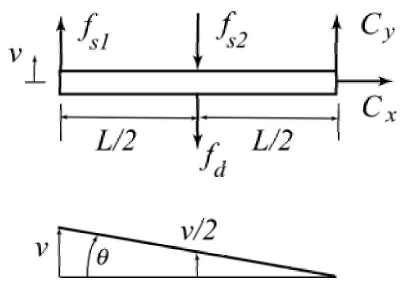

Figure 2

$$\circlearrowleft + \sum M_C = I_C\ddot{\theta}$$

where

$$I_C = \frac{1}{3}ML^2$$

From the free-body diagram (Fig. 2 top),

$$f_{s1}L - (f_{s2} + f_d)\frac{L}{2} = I_C\ddot{\theta}$$

The constitutive relationships of the linear springs and the linear dashpot are:

$$f_{s1} = k_1[z(t) - v], \quad f_{s2} = k_2\frac{v}{2}, \quad f_d = c\frac{\dot{v}}{2}$$

From the deformation diagram (Fig. 2 bottom) and kinematics, for small θ,

$$v = L\theta, \quad \text{or} \quad \theta = \frac{v}{L}$$

Combining the above equations, we get

$$k_1[z(t) - v]L - \left(k_2\frac{v}{2} + c\frac{\dot{v}}{2}\right)\frac{L}{2} = \frac{1}{3}ML^2\left(\frac{\ddot{v}}{L}\right)$$

Finally, the equation of motion of the beam in terms of displacement v is

$$4M\ddot{v} + 3c\dot{v} + (12k_1 + 3k_2)v = 12k_1z(t)$$ **Ans. (a)**

(b) Determine the equation of motion of the beam in terms of the spring force f_{s1}.

In the above answer to Part (a), substitute the force-deformation of spring 1 in the form

$$v = z(t) - \frac{f_{s1}}{k_1}$$

Then, the equation of motion for f_{s1} is

$$4M\ddot{f}_{s1} + 3c\dot{f}_{s1} + (12k_1 + 3k_2)f_{s1} = k_1(4M\ddot{z} + 3c\dot{z} + 3k_2z)$$ **Ans. (b)**

Although the left-hand sides of these two equations of motion have exactly the same form, the right-hand side of Ans. (b) is more complicated.

2.5 Solution

Determine the equation of motion of the beam in Fig. 1 in terms of displacement v.

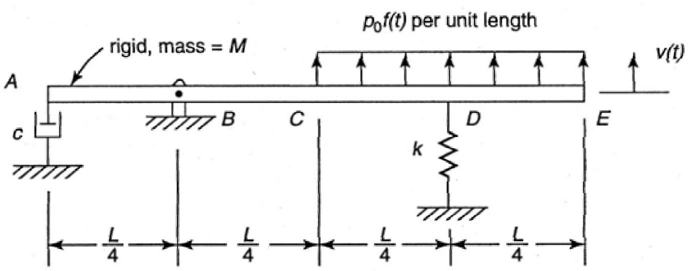

Figure 1

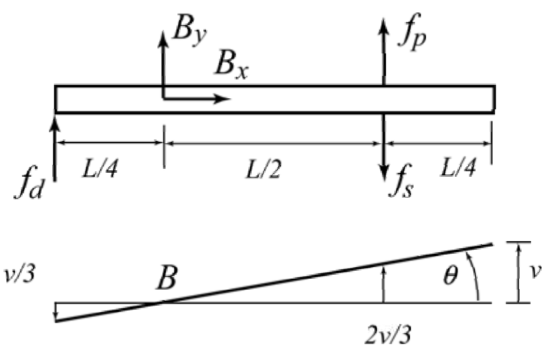

Figure 2

For the equation of motion, take

$$\circlearrowleft + \sum M_B = I_B\ddot{\theta}$$

where

$$I_B = \frac{1}{3}\left(\frac{M}{4}\right)\left(\frac{L}{4}\right)^2 + \frac{1}{3}\left(\frac{3M}{4}\right)\left(\frac{3L}{4}\right)^2 = \frac{7ML^2}{48}$$

From the free-body diagram (Fig. 2 top),

$$-f_d\frac{L}{4} + f_p\frac{L}{2} - f_s\frac{L}{2} = I_B\ddot{\theta}$$

where

$$f_p = \frac{p_0L}{2}f(t)$$

From the deformation diagram (Fig. 2 bottom) and kinematics, for small θ,

$$v = \frac{3L}{4}\theta, \quad \text{or} \quad \theta = \frac{4v}{3L}$$

The constitutive relationships for the linear spring and the linear dashpot are:

$$f_s = k\frac{2v}{3}, \quad f_d = c\frac{\dot{v}}{3}$$

For the right-hand side of the EOM, we get

$$I_B\ddot{\theta} = \frac{7ML^2}{48}\left(\frac{4\ddot{v}}{3L}\right) = \frac{7}{36}ML\ddot{v}$$

Combining the above equations, we get the final EOM in the desired form.

$$7M\ddot{v} + 3c\dot{v} + 12kv = 9p_0Lf(t)$$ **Ans.**

2.7 Solution

Use the *Principle of Virtual Displacements* to derive the equation of motion of the beam in Fig. 1. Let $v(t)$, the vertical displacement at end B, be the displacement coordinate.

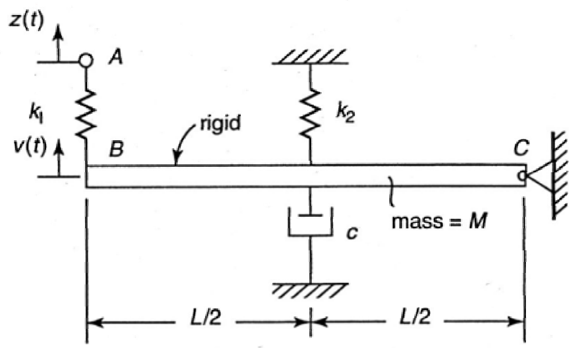

Figure 1

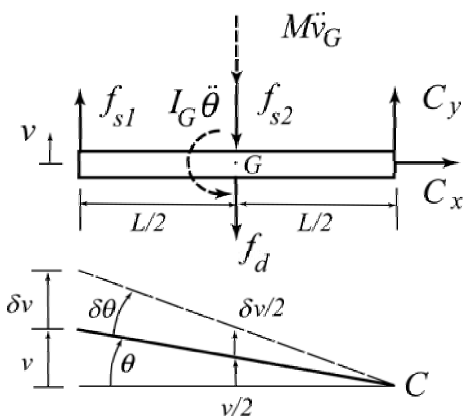

Figure 2

From Eq. 2.25,

$\delta\mathcal{W}^* \equiv \delta\mathcal{W}_{\text{real}} + \delta\mathcal{W}_{\text{inertia}} = 0$

From the deformation diagram (Fig. 2 bottom) and kinematics of small-angle rotation of the beam,

$v = L\theta, \quad \delta v = L\delta\theta, \quad v_G = v/2$

From dynamics,

$I_G = \dfrac{1}{12}ML^2$

The constitutive relationships of the linear springs and the linear dashpot are:

$f_{s1} = k_1[z(t) - v], \quad f_{s2} = k_2\dfrac{v}{2}, \quad f_d = c\dfrac{\dot{v}}{2}$

The terms in the virtual-work equation are:

$$\delta\mathcal{W}_{\text{real}} + \delta\mathcal{W}_{\text{inertia}} = f_{s1}\delta v - f_{s2}\frac{\delta v}{2} - f_d\frac{\delta v}{2} - \frac{M\ddot{v}}{2}\frac{\delta v}{2} - I_G\ddot{\theta}\,\delta\theta = 0$$

Finally, substituting for I_G, θ, and $\delta\theta$ and obtaining the form $[\,\cdot\,]\delta v = 0$, we take the expression in brackets as the equation of motion.

$4M\ddot{v} + 3c\dot{v} + (12k_1 + 3k_2)v = 12k_1 z(t)$ **Ans.**

2.9 Solution

Use the *Principle of Virtual Displacements* to derive the equation of motion of the beam in Fig. 1. Let the angular displacement θ (counterclockwise positive) be the displacement coordinate.

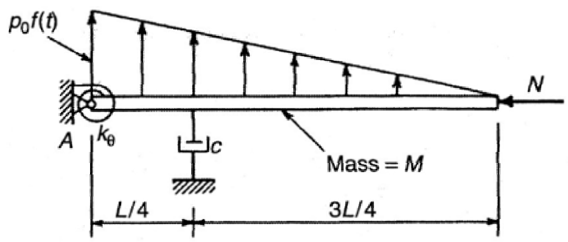

Figure 1

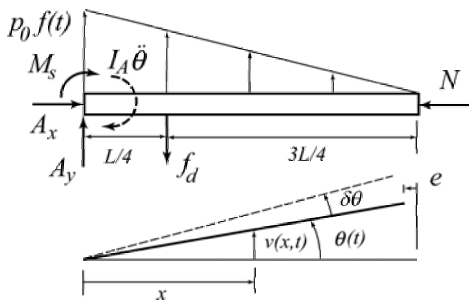

Figure 2

From Eq. 2.25,

$$\delta\mathcal{W}^* \equiv \delta\mathcal{W}_{\text{real}} + \delta\mathcal{W}_{\text{inertia}} = 0$$

From the deformation diagram (Fig. 2 bottom) and kinematics of small-angle rotation of the beam,

$$e = L(1-\cos\theta) \approx \frac{1}{2}L\theta^2, \quad \delta e = L\theta\delta\theta$$

$$v(x,t) = \frac{x}{L}\,\theta(t), \quad \delta v(x) = \frac{x}{L}\,\delta\theta$$

The constitutive relationships of the linearly elastic spring and the linear viscous dashpot are:

$$M_s = k_\theta\theta, \quad f_d = c\,\frac{L}{4}\dot{\theta}$$

The virtual work terms are:

$$\begin{aligned}\delta\mathcal{W}_{\text{real}} &= \delta\mathcal{W}_p + \delta\mathcal{W}_s + \delta\mathcal{W}_d + \delta\mathcal{W}_N \\ &= \int_0^L \left[p_0 f(t)\left(1-\frac{x}{L}\right)\right]\delta v(x)dx - M_s\delta\theta - f_d\left(\frac{L}{4}\,\delta\theta\right) + N\delta e\end{aligned}$$

$$\delta\mathcal{W}_{\text{inertia}} = -I_A\ddot{\theta}\,\delta\theta$$

Finally, substituting for $\delta v(x)$ and δe and obtaining the form $[\,\cdot\,]\,\delta\theta = 0$, we take the expression in brackets as the equation of motion.

$$I_A\ddot{\theta} + c\left(\frac{L^2}{6}\right)\dot{\theta} + (k_\theta - NL)\,\theta = p_0\left(\frac{L^2}{6}\right)f(t) \qquad \textbf{Ans.}$$

where, from dynamics,

$$I_A = \frac{1}{3}ML^2$$

2.11 Solution

Use the *Principle of Virtual Displacements* to derive the equation of motion of the thin, uniform bar of weight W that rotates through a small angle about the (vertical) z axis.

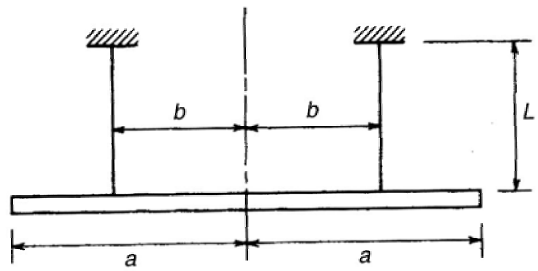

Figure 1

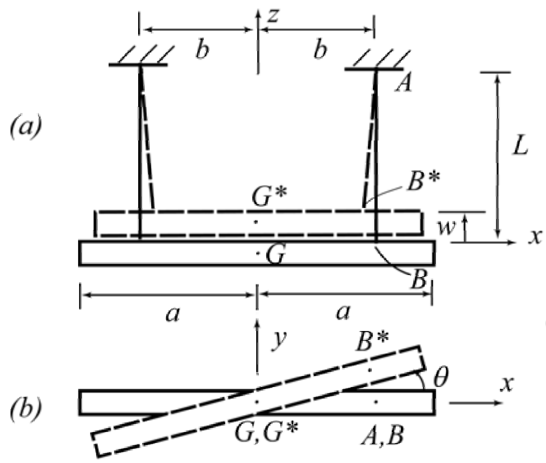

Figure 2 Deformation geometry.

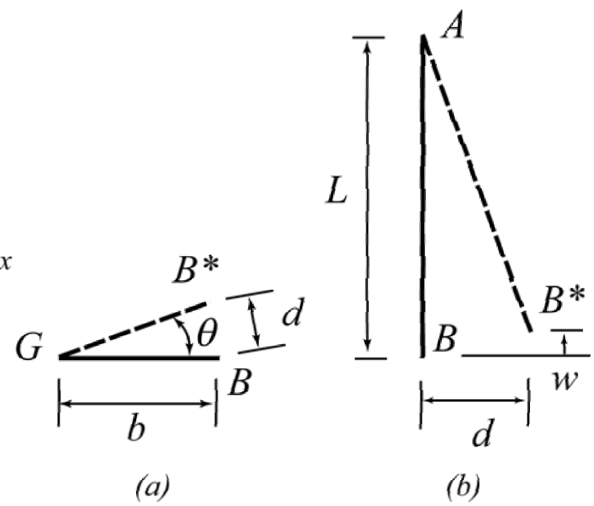

Figure 3 More deformation geometry.

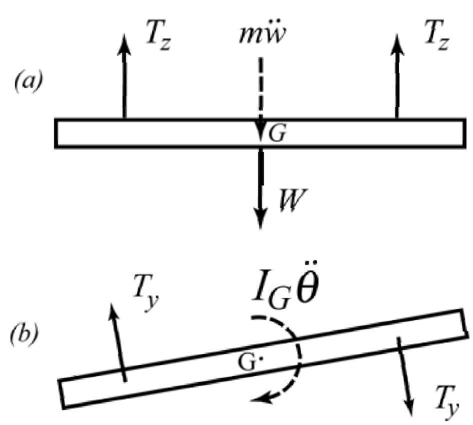

Figure 4 Force Diagrams.

Problem 2.11 (p. 2)

We first examine the geometry of deformation in order to relate the vertical displacement of mass center G (and of B) to the rotation angle θ. For small angle θ, from Fig. 3a,

$$d = b\theta$$

From Fig. 3b,

$$w = L - \sqrt{L^2 - d^2} = L - L\sqrt{1 - (d/L)^2} \approx L - L\left[1 - \frac{1}{2}\left(\frac{d}{L}\right)^2\right] = \frac{1}{2}\left(\frac{b^2}{L}\right)\theta^2$$

Therefore, the vertical displacement w is second-order in θ, and since θ is considered to be small, the vertical inertia term in Fig. 4a will be neglected. From the vertical equation of motion based on Fig. 4a,

$$T_z = \frac{W}{2}$$

Since the tension T in the support wires when the bar is at angle θ will be along the direction $B^* \to A$ in Fig. 3b, the relationship between the force components T_y and T_z in Fig. 4 is

$$\frac{T_y}{T_z} = \frac{d}{L - w} \approx \frac{d}{L}$$

Therefore,

$$T_y = \frac{d}{L}T_z = \frac{bW}{2L}\,\theta$$

From Eq. 2.25, the principle of virtual work is

$$\delta\mathcal{W}^* \equiv \delta\mathcal{W}_{\text{real}} + \delta\mathcal{W}_{\text{inertia}} = 0$$

Then, substituting the forces that are shown in Fig. 4b into the principle of virtual work applied to rotation of the bar, we get the following equation of motion:

$$-(I_G\ddot{\theta})\,\delta\theta - 2T_y(b\,\delta\theta) = 0$$

$$\frac{1}{12}\left(\frac{W}{g}\right)(2a)^2\ddot{\theta} + \frac{b^2W}{L}\,\theta = 0$$

or

$$\ddot{\theta} + \left(\frac{3b^2g}{a^2L}\right)\theta = 0 \qquad \textbf{Ans.}$$

2.13 Solution

Use the *Assumed-Modes Method* to derive the equation of motion for axial motion of the bar with tip mass, as shown in Fig. 1. Let $\psi(x) = x/L$ be the shape function.

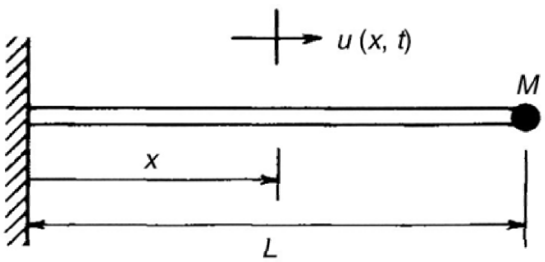

Figure 1

The axial displacement is assumed to have the form

$u(x,t) = \psi(x)u(t)$

Since there are no external forces acting on the bar, the equation of motion will have the form

$m_u\ddot{q}_u + k_u q_u = 0$

where

$$m_u = \int_0^L \rho A\psi^2\,dx + M\psi^2(L), \quad k_u = \int_0^L AE(\psi')^2\,dx$$

Then,

$$m_u = \int_0^L \rho A\psi^2\,dx + M[\psi(L)]^2 = \frac{\rho A}{L^2}\int_0^L x^2\,dx + M(1)^2 = \frac{\rho AL}{3} + M$$

$$k_u = \int_0^L AE(\psi')^2\,dx = \frac{AE}{L^2}\int_0^L dx = \frac{AE}{L}$$

Finally, the equation of motion for the SDOF assumed-modes model is

$$\left(\frac{\rho AL}{3} + M\right)\ddot{u} + \left(\frac{AE}{L}\right)u = 0$$ **Ans.**

2.15 Solution

Use the *Assumed-Modes Method* to derive the equation of motion for axial motion of the tapered bar in Fig. 1. Let $\psi(x) = \sin(\pi x/2L)$ be the shape function.

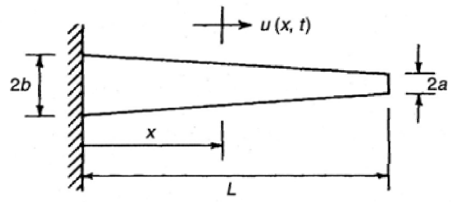

Figure 1

Since the bar is flat,

$$A(x) = hw(x) = h\left[2b - \frac{x}{L}(2b - 2a)\right]$$

Since there are no external forces acting on the bar, the equation of motion will have the form

$$m_u \ddot{q}_u + k_u q_u = 0$$

where

$$m_u = \int_0^L \rho A(x)\psi^2\,dx, \quad k_u = \int_0^L A(x)E(\psi')^2\,dx$$

Therefore,

$$m_u = \int_0^L \rho A(x)\psi^2\,dx = \rho h \int_0^L \left[2b - \frac{x}{L}(2b-2a)\right]\sin^2\left(\frac{\pi x}{2L}\right)dx$$

$$= c_1 \int_0^L \sin^2\left(\frac{\pi x}{2L}\right)dx + c_2 \int_0^L x\sin^2\left(\frac{\pi x}{2L}\right)dx$$

$$k_u = \int_0^L A(x)E(\psi')^2\,dx = Eh\left(\frac{\pi}{2L}\right)^2 \int_0^L \left[2b - \frac{x}{L}(2b-2a)\right]\cos^2\left(\frac{\pi x}{2L}\right)dx$$

$$= c_3 \int_0^L \cos^2\left(\frac{\pi x}{2L}\right)dx + c_4 \int_0^L x\cos^2\left(\frac{\pi x}{2L}\right)dx$$

Finally, the equation of motion for the SDOF assumed-modes model is

$$\left.\begin{aligned} & m_u \ddot{q}_u + k_u q_u = 0\ , \quad \text{where} \\ & m_u = \rho bhL - \rho hL(2b-2a)\left(\frac{1}{4} + \frac{1}{\pi^2}\right) \\ & k_u = \frac{\pi^2 bhE}{4L} - \frac{\pi^2(2b-2a)hE}{L}\left(\frac{1}{4} - \frac{1}{\pi^2}\right) \end{aligned}\right\}$$

Ans.

2.17 Solution

Use the *Assumed-Modes Method* to derive the equation of motion for axial vibration of the circular rod that is fixed at both ends. See Fig. 1. Let $\psi(x) = (x/L)(1 - x/L)$ be the shape function.

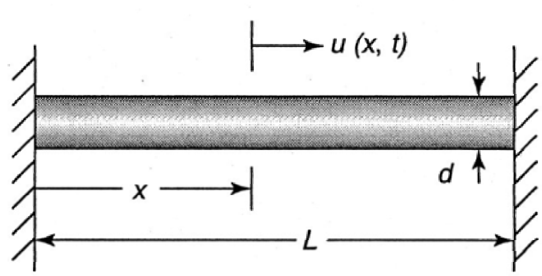

Figure 1

Since there are no external forces acting on the bar, the equation of motion will have the form

$m_u \ddot{q}_u + k_u q_u = 0$

where

$$m_u = \int_0^L \rho A \psi^2 \, dx, \quad k_u = \int_0^L AE(\psi')^2 \, dx$$

where

$$\psi(x) = \frac{x}{L} - \left(\frac{x}{L}\right)^2, \quad \psi(x)' = \frac{1}{L}\left(1 - 2\frac{x}{L}\right)$$

which satisfies the two fixed-end requirements: $\psi(0) = \psi(L) = 0$. Therefore,

$$m_u = \int_0^L \rho A \psi^2 \, dx = \rho A \int_0^L \left[\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right] dx = \rho AL\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= \frac{\rho AL}{30}$$

$$k_u = \int_0^L AE(\psi')^2 \, dx = \frac{AE}{L^2} \int_0^L \left[1 - 4\left(\frac{x}{L}\right) + 4\left(\frac{x}{L}\right)^2\right] dx = \frac{AE}{L}\left(1 - 2 + \frac{4}{3}\right)$$

$$= \frac{AE}{3L}$$

where $A = \pi d^2/4$. Finally, the equation of motion for the SDOF assumed-modes model is

$$\left(\frac{\rho AL}{30}\right)\ddot{q}_u + \left(\frac{AE}{3L}\right)q_u = 0$$ **Ans.**

2.19 Solution

Use the *Assumed-Modes Method* to derive the equation of motion for transverse vibration of the cantilever beam in Fig. 1. Use a simple polynomial shape function.

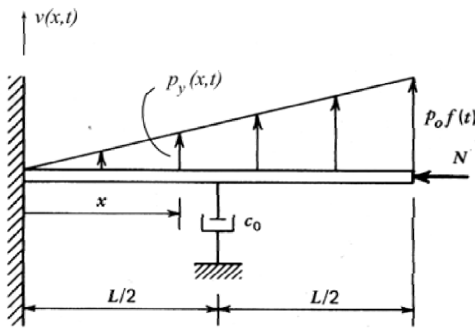

Figure 1

Let the shape function be the simplest polynomial that satisfies the fixed-end condition at the left end, that is $\psi(0) = \psi'(0) = 0$. Pick

$$\psi(x) = \left(\frac{x}{L}\right)^2$$

Then, the transverse displacement is approximated by

$$v(x,t) = \psi(x)q_v(t) = \left(\frac{x}{L}\right)^2 q_v(t)$$

Since the beam has a time-dependent transverse load and a constant axial compressive load, the equation of motion will have the form

$$m_v\ddot{q}_v + c_v\dot{q}_v + (k_v - k_G)q_v = p_v(t)$$

where

$$m_v = \int_0^L \rho A\psi^2\, dx = \frac{\rho A}{L^4}\int_0^L x^4\, dx = \frac{\rho AL}{5}$$

$$c_v = c_0\,[\psi(L/2)]^2 = c_0\left[\left(\frac{1}{2}\right)^2\right]^2 = \frac{c_0}{16}$$

$$k_v = \int_0^L EI(\psi'')^2\, dx = EI\left(\frac{2}{L^2}\right)^2\int_0^L dx = \frac{4EI}{L^3}$$

$$k_G = N\int_0^L (\psi')^2\, dx = N\left(\frac{2}{L^2}\right)^2\int_0^L x^2\, dx = \frac{4N}{3L}$$

$$p_v(t) = \int_0^L p_v(x,t)\psi(x)\, dx = \int_0^L \left[p_0 f(t)\left(\frac{x}{L}\right)\right]\left(\frac{x}{L}\right)^2 dx = \frac{p_0 L}{4} f(t)$$

Therefore, the final equation of motion is

$$\frac{\rho AL}{5}\ddot{q}_v + \frac{c_0}{16}\dot{q}_v + \left(\frac{4EI}{L^3} - \frac{4N}{3L}\right)q_v = \frac{1}{4}p_0 L f(t) \qquad \textbf{Ans.}$$