CHAPTER 1

FOUNDATIONS OF HEAT TRANSFER

PROB. 1.1: Since the flow is steady, the first law (1.31) reduces to

$$\int_{c.s.} \rho \left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = q_{c.s.} - \dot{W}_{shear} - \dot{W}_{shaft} \int_{c.v.} \dot{q}_e\, d\vartheta \qquad (1)$$

Assume that all the properties are constant over cross-sections ① and ②.

Thus,

$$\int_{c.s.} \rho\left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = \rho_2\left(h_2+\frac{1}{2}V_2^2+gz_2\right)V_2A_2 - \rho_1\left(h_1+\frac{1}{2}V_1^2+gz_1\right)V_1A_1$$

The continuity condition yields

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m}$$

Thus,

$$\int_{c.s.} \rho\left(h+\frac{1}{2}V^2+gz\right)\vec{V}\cdot\hat{n}\,dA = \dot{m}\left[\left(h_2+\frac{1}{2}V_2^2+gz_2\right)-\left(h_1+\frac{1}{2}V_1^2+gz_1\right)\right]$$

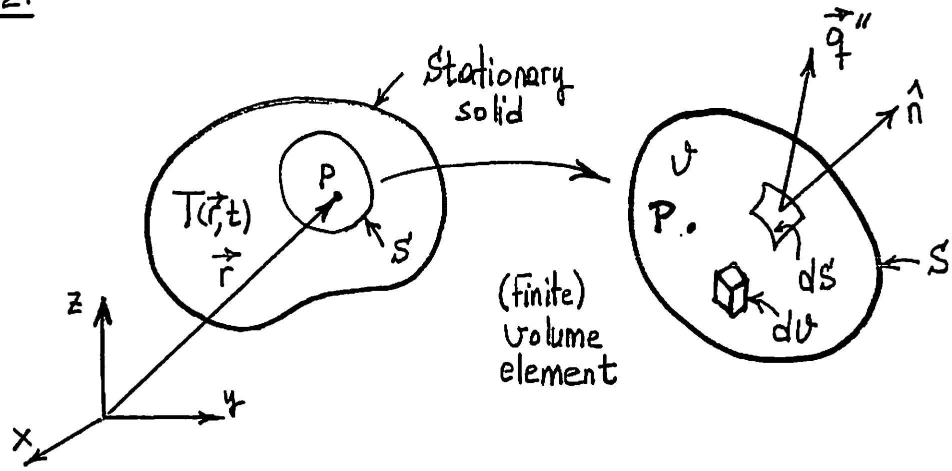
Furthermore, if it is further assumed that the flow is frictionless, then $\dot{W}_{shear}=0$. In addition,

$$q_{c.s.} = q\ ,\quad \dot{W}_{shaft} = \dot{W} \quad \text{and} \quad \dot{q}_e = 0$$

Therefore, the steady-flow energy equation (1) can be rewritten as

$$\boxed{h_1+\frac{1}{2}V_1^2+gz_1+\frac{q}{\dot{m}} = h_2+\frac{1}{2}V_2^2+gz_2+\frac{\dot{W}}{\dot{m}}}$$

PROB. 1.2:



Since $\vec{V}=0$, $\dot{W}_{shear}=\dot{W}_{shaft}=0$, Eq.(1.30) reduces to

$$\underbrace{\int_{\mathcal{V}} \rho \frac{\partial e}{\partial t} d\mathcal{V}} = q_s + \int_{\mathcal{V}} \dot{q}_e d\mathcal{V} \qquad (1)$$

$$\int_{\mathcal{V}} \rho \frac{\partial u}{\partial t} d\mathcal{V} = \int_{\mathcal{V}} \rho c \frac{\partial T}{\partial t} d\mathcal{V} \qquad (2)$$

for solids $du = c\,dT$, c = specific heat

Furthermore,

$$q_s = \int_S \vec{q}.\hat{n}\, dS = \int_{\mathcal{V}} \vec{\nabla}.q'' d\mathcal{V} \qquad (3)$$

Here, $\vec{q}''$ is the heat flux vector at dS, where $\hat{n}$ represents the outward-drawn unit vector normal to dS. Then, by substituting Eqs.(2) and (3) into (1), we obtain

$$\int_{\mathcal{V}} \left[-\vec{\nabla}.\vec{q}'' + \dot{q}_e - \rho c \frac{\partial T}{\partial t}\right] d\mathcal{V} = 0$$

Since this volume integral vanishes for every volume element $\mathcal{V}$, its integrand must vanish everywhere, thus yielding

$$\boxed{-\vec{\nabla}\cdot\vec{q}'' + \dot{q}_e = \rho c \frac{\partial T}{\partial t}}$$

PROB.1.3: Under the conditions stated the 1st law gives:

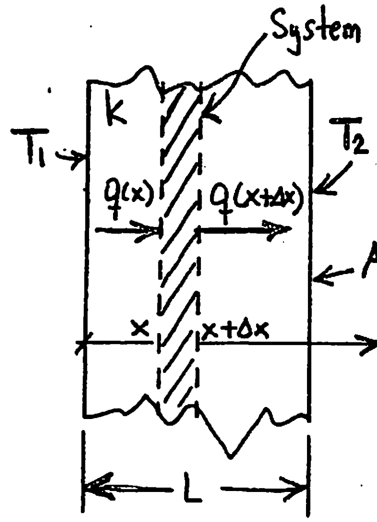
$$q(x) = q(x+\Delta x)$$

As $\Delta x \to 0$,

$$q(x+\Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

Hence, as $\Delta x \to 0$, the first law reduces to

$$\frac{dq}{dx} = 0 \quad \text{or} \quad q(x) = \text{const.}$$



PROB.1.4:

$$\underset{\uparrow \atop \text{First law}}{\frac{dq}{dx} = 0} \quad \& \quad \underset{\uparrow \atop \text{Fourier's law}}{q = -kA\frac{dT}{dx}} \quad \Rightarrow \quad \underbrace{\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) = 0}_{\frac{d^2T}{dx^2} = 0}$$

$$\frac{d^2T}{dx^2} = 0 \quad \Rightarrow \quad T(x) = Ax + B$$

$$\left.\begin{aligned} T(0) &= T_1 \quad \Rightarrow \quad B = T_1 \\ T(L) &= T_2 \quad \Rightarrow \quad A = \frac{T_2 - T_1}{L} \end{aligned}\right\} \qquad \therefore \ \boxed{T(x) = T_1 - \frac{T_1 - T_2}{L}x}$$

PROB.1.5: The first law gives (see Prob.1.3)

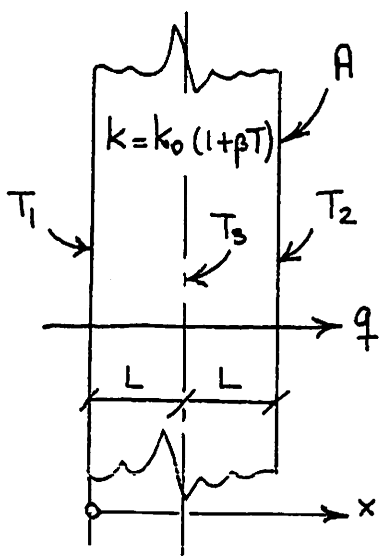
$$q = -kA\frac{dT}{dx} = \text{Const.} \Rightarrow k\frac{dT}{dx} = -\frac{q}{A}$$

$$\text{or} \quad k_0(1+\beta T)dT = -\frac{q}{A}dx$$

$$\therefore [T(x_1) - T(x_2)] + \beta\frac{1}{2}[T^2(x_1) - T^2(x_2)] = -\frac{q}{Ak_0}(x_1 - x_2)$$

$$T_3 - T_1 + \frac{\beta}{2}(T_3^2 - T_1^2) = -\frac{qL}{Ak_0} \quad \text{— (1)}$$

$$T_2 - T_3 + \frac{\beta}{2}(T_2^2 - T_3^2) = -\frac{qL}{Ak_0} \quad \text{— (2)}$$

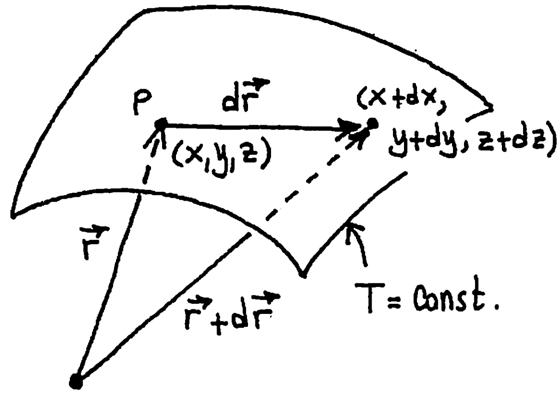


These two equations yield:

$$\left.\begin{aligned} \beta &= -2\,\frac{T_1+T_2-2T_3}{T_1^2+T_2^2-2T_3^2} \\ k_0 &= \frac{2(q/A)\times L}{(T_2-T_1)\left[1+\frac{\beta}{2}(T_1+T_2)\right]} \end{aligned}\right\} \Rightarrow k(T)=k_0[1+\beta T]$$

<u>PROB. 1.6:</u>

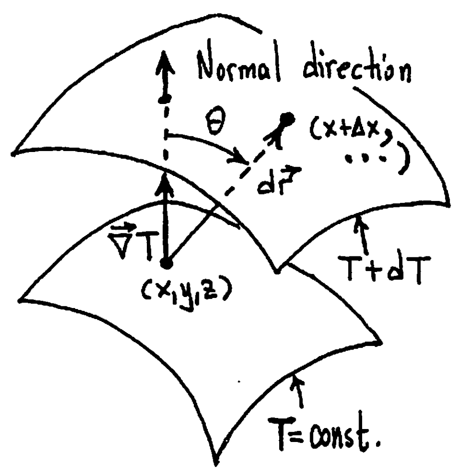
$$\begin{aligned} dT &= \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \\ &= \vec{\nabla}T \cdot \underbrace{(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)}_{d\vec{r}} \\ &= \vec{\nabla}T \cdot d\vec{r} \end{aligned}$$



On the T = Const. surface $dT=0$. Thus, $\vec{\nabla}T$ is a vector normal to <u>all</u> such $d\vec{r}$. On the other hand, $d\vec{r}$ is a vector tangent to the isothermal surface passing through P. Therefore, $\vec{\nabla}T$ is a vector normal to the isothermal surface.

Again, $dT = \vec{\nabla}T \cdot d\vec{r}$

In order for $dT>0$, $\cos\theta>0$; that is, $\vec{\nabla}T$ has to point in the direction of increasing temperature as shown.



PROB. 1.7:

$$\vec{\nabla} T \cdot \hat{s} = \left(\hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}\right)\cdot\left(\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma\right)$$

Direction Cosines of $\hat{s}$

$$= \frac{\partial T}{\partial x}\underbrace{\cos\alpha}_{\frac{dx}{ds}} + \frac{\partial T}{\partial y}\underbrace{\cos\beta}_{\frac{dy}{ds}} + \frac{\partial T}{\partial z}\underbrace{\cos\gamma}_{\frac{dz}{ds}}$$

$$= \frac{\partial T}{\partial s}$$

PROB. 1.8:

$$T\,(^\circ C) = 150 - 400\,x^2$$

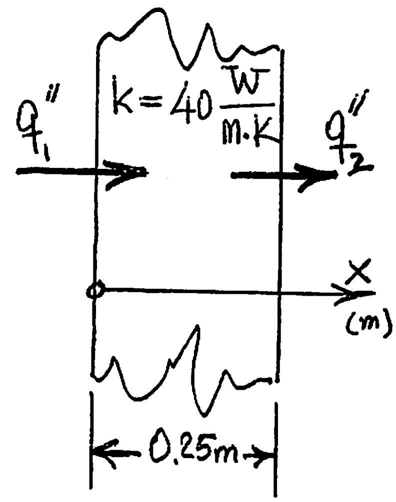
$$q_1'' = -k\left(\frac{\partial T}{dx}\right)_{x=0} = -k\,(-800x)_{x=0}$$

$$= 0\ W/m^2$$

$$q_2'' = -k\left(\frac{dT}{dx}\right)_{x=L} = -k\,(-800\,L)$$

$$= 40 \times 800 \times 0.25$$

$$= 8{,}000\ W/m^2$$



PROB. 1.9:

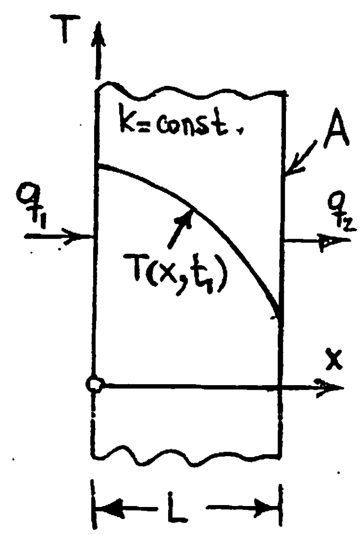
$$q_1 = -kA\left(\frac{\partial T}{\partial x}\right)_{x=0} \quad \text{and} \quad q_2 = -kA\left(\frac{\partial T}{\partial x}\right)_{x=L}$$

$$-\left(\frac{\partial T}{\partial x}\right)_{x=0} < -\left(\frac{\partial T}{\partial x}\right)_{x=L}$$

$$\Downarrow$$

$$q_1 < q_2$$

$\therefore$ The wall is being cooled at t_1.



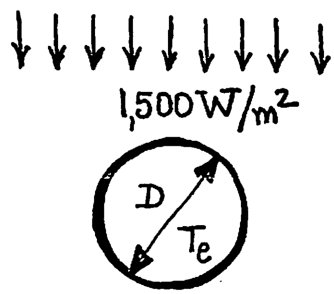
PROB.1.10:

$$h = \frac{-k\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_W - T_\infty} = \frac{-(0.62)\,\frac{W}{m\cdot K} \times (-80 \times 800)\,\frac{K}{m}}{(100-20)\,K}$$

$$= 496\ \frac{W}{m^2\cdot K}$$

PROB.1.11:

Energy balance yields



L = Length of cylinder

$$\alpha_s\,(1{,}500)\,\frac{W}{m^2} \times (D \times L)\ m^2$$

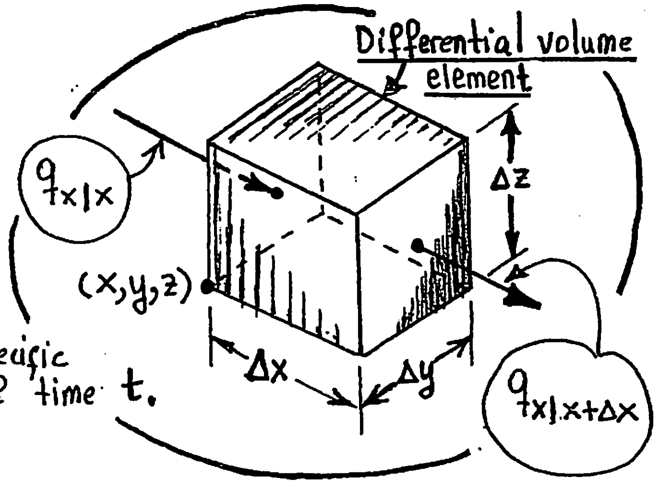
$$= \varepsilon\,(\pi D L)\ m^2 \times \underset{\substack{\uparrow \\ 5.67\times10^{-8}\ W/(m^2\cdot K^4)}}{\sigma}\,(T_e^4)\ K^4$$

$$T_e = \frac{1500}{\pi \times 5.67\times10^{-8}} = 303K = 30^\circ C.$$

CHAPTER 2

GENERAL HEAT CONDUCTION EQUATION

PROB. 2.1:



- Homogeneous and isotropic solid, i.e., $k = \text{const.}$
- Let there be no heat sources or sinks, i.e., $\dot{q} = 0$.
- Assume the density ρ and specific heat c are independent of time t.

$$\left\{\begin{matrix}\text{Net rate of heat conduction} \\ \text{into the C.V. in the} \\ x\text{- direction}\end{matrix}\right\} = q_{x|x} - q_{x|x+\Delta x} = q_{x|\Delta x} - \left\{q_{x|x} + \frac{\partial q_{x|x}}{\partial x}\Delta x\right\}$$

$$= -\frac{\partial q_{x|x}}{\partial x}\Delta x$$

where

$$q_{x|x} = -k\,(\Delta y \Delta z)\frac{\partial T}{\partial x} \leftarrow \underline{\text{Rate of heat conduction in the } x\text{-direction at } x \text{ into the C.V.}}$$

$$\therefore\; q_{x|x} - q_{x|x+\Delta x} = k\frac{\partial^2 T}{\partial x^2}\Delta V, \quad \Delta V = \Delta x \Delta y \Delta z$$

Similarly,

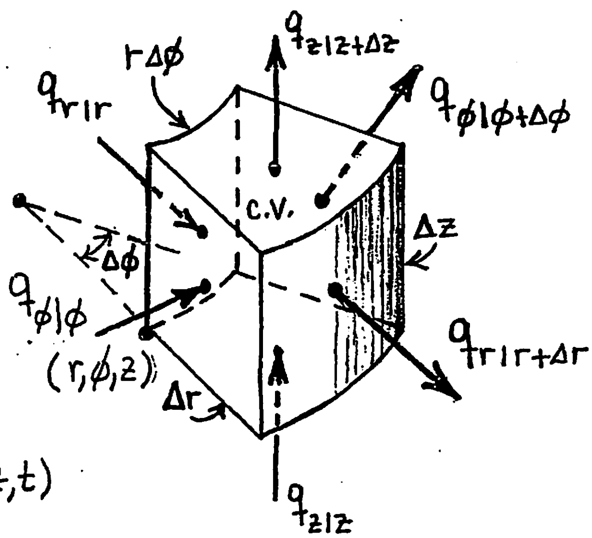
$$q_{y|y} - q_{y|y+\Delta y} = k\frac{\partial^2 T}{\partial y^2}\Delta V \quad \text{and} \quad q_{z|z} - q_{z|z+\Delta z} = k\frac{\partial^2 T}{\partial z^2}\Delta V$$

Thus, energy balance on the C.V. gives

$$k\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right\}\cancel{\Delta V} = \left\{\begin{matrix}\text{Net rate of increase} \\ \text{of internal energy} \\ \text{in C.V.}\end{matrix}\right\} = \frac{\partial}{\partial t}\left\{Tc\rho\cancel{\Delta V}\right\}$$

$$\therefore\; \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}, \quad \alpha = \frac{k}{\rho c}$$

PROB. 2.2: (a)



$T = T(r, \phi, z, t)$

$\Delta V = r\,\Delta r\,\Delta\phi\,\Delta z$

$$q_{r|r} - q_{r|r+\Delta r} = -\frac{\partial q_{r|r}}{\partial r}\Delta r = \frac{\partial}{\partial r}\left[k\, r\Delta\phi\Delta z \frac{\partial T}{\partial r}\right]\Delta r = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\Delta V$$

$$q_{\phi|\phi} - q_{\phi|\phi+\Delta\phi} = -\frac{\partial q_{\phi|\phi}}{\partial (r\phi)} r\Delta\phi = \frac{\partial}{\partial \phi}\left[k\Delta r\Delta z \frac{\partial T}{\partial (r\phi)}\right]\Delta\phi = \frac{k}{r^2}\frac{\partial^2 T}{\partial \phi^2}\Delta V$$

$$q_{z|z} - q_{z|z+\Delta z} = -\frac{\partial q_{z|z}}{\partial z}\Delta z = \frac{\partial}{\partial z}\left[k\, r\Delta\phi\Delta r \frac{\partial T}{\partial z}\right]\Delta z = k\frac{\partial^2 T}{\partial z^2}\Delta V$$

$$\therefore\ \underbrace{k\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right\}\cancel{\Delta V}}_{\text{Net rate of heat conduction into c.v.}} = \underbrace{\frac{\partial}{\partial t}\left\{Tc\rho\cancel{\Delta V}\right\}}_{\text{Net rate of increase of internal energy in c.v.}}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}},\quad \alpha = \frac{k}{\rho c}$$

PROB. 2.3: General heat conduction equation in the rectangular coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

a) The cylindrical coordinates:

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}\right\} \quad \begin{aligned} r &= (x^2+y^2)^{1/2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$

Thus,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} = \cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \phi}{\partial x}$$

$$= \cos^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi} - 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial \phi}$$

$$+ \frac{\sin^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

Similarly,

$$\frac{\partial T}{\partial y} = \sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial \phi}$$

$$\frac{\partial^2 T}{\partial y^2} = \sin^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial \phi\partial r} - 2\frac{\sin\phi\cos\phi}{r^2}\frac{\partial T}{\partial \phi}$$

$$+ \frac{\cos^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

In addition, in both coordinate systems: $\frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial z^2}$

Therefore,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

and the general heat conduction equation in the cylindrical coordinate is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

b) The spherical coordinates:

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ \theta &= \cos^{-1}\frac{z}{[x^2+y^2+z^2]^{1/2}} \end{aligned}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \sin\theta\cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} + \frac{\cos\theta\cos\phi}{r}\frac{\partial T}{\partial \theta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 T}{\partial r^2} + 2\frac{\cos\phi\sin\phi}{r^2\sin^2\theta}\frac{\partial T}{\partial \phi} - 2\frac{\sin\phi\cos\phi}{r}\frac{\partial^2 T}{\partial r\partial\phi}$$

$$+ \frac{\cos\theta}{r^2}\left(-2\sin\theta\cos^2\phi + \frac{\sin^2\phi}{\sin\theta}\right)\frac{\partial T}{\partial \theta} + 2\frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 T}{\partial r\partial\theta}$$

$$+ \left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right)\frac{\partial T}{\partial r} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

$$- 2\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 T}{\partial\phi\partial\theta} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 T}{\partial \theta^2}$$

Similarly,

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \sin\theta\sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} + \frac{\cos\theta\sin\phi}{r}\frac{\partial T}{\partial \theta}$$