Problem 2.1

Starting from Eq. (2.22), show that for a parallelflow heat exchanger, Eq. (2.26a) becomes

$$\frac{T_{h_2} - T_{c_2}}{T_{h_1} - T_{c_1}} = \exp\left[-\left(\frac{1}{C_h} + \frac{1}{C_c}\right)UA\right]$$

SOLUTION:

The heat transferred across the area dA is:

$$\delta Q = U(T_h - T_c)dA \qquad (1)$$

The heat transfer rate can also be written as the change in enthalpy of each fluid (with the correct sign) between the area A and A+dA:

* for the hot fluid ($dT_h<0$)

$$\delta Q = -\dot{m}_h c_{p,h} dT_h \qquad (2)$$

* for the cold fluid ($dT_c>0$)

$$\delta Q = \dot{m}_c c_{p,c} dT_c \qquad (3)$$

The notion of heat capacity can be introduced as:

$$C = \dot{m} c_p \qquad (4)$$

This parameter represents the rate of heat transferred by a fluid when its temperature varies with one degree.

The equation (2) and (3) give:

$$\delta Q = -C_h dT_h = C_c dT_c \qquad (5)$$

Equations (1) and (5) give:

$$\frac{dT_h}{T_h - T_c} = -\frac{U}{C_h} dA \qquad (6)$$

$$\frac{dT_c}{T_h - T_c} = -\frac{U}{C_c} dA \qquad (7)$$

Subtracting equation (7) from (6):

$$\frac{d(T_h - T_c)}{T_h - T_c} = \left(\frac{1}{C_c} - \frac{1}{C_h}\right) UdA \qquad (8)$$

Considering the overall heat transfer coefficient U=constant, equation (8) can be integrated:

$$\ln(T_h - T_c) = \left(\frac{1}{C_c} - \frac{1}{C_h}\right) UA + \ln B \qquad (9)$$

$$T_h - T_c = B\exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right) UA\right] \qquad (10)$$

The constant of integration, K is obtained from the boundary condition at the inlet:

at A=0, $T_h - T_c = T_{h1} - T_{c2}$ (11)

$$K = T_{h1} - T_{c2} \quad (12)$$

Introducing equation (12) in (10) we have:

$$\frac{T_h - T_c}{T_{h1} - T_{c2}} = \exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] \quad (13)$$

At the outlet the heat transfer area is A_t=A and T_h-T_c=T_{h2}-T_{c2} and:

$$\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}} = e^{-\left(\frac{1}{C_h} + \frac{1}{C_c}\right)UA} \quad (14)$$

Problem 2.2

Show that for a parallel flow heat exchanger the variation of the hot fluid temperature along the heat exchanger is given by

$$\frac{T_h - T_{h1}}{T_{h1} - T_{c1}} = \frac{-C_c}{C_h + C_c}\left\{1 - e^{-\left(\frac{1}{C_h}+\frac{1}{C_c}\right)UA}\right\}$$

Obtain a similar expression for the variation of the cold fluid temperature along the heat exchanger. Also show that for $A \to \infty$, the temperature will be equal to mixing-cup temperature of the fluids which is given by

$$T_\infty = \frac{C_h T_{h1} + C_c T_{c1}}{C_h + C_c}$$

SOLUTION:

The heat transferred across the area dA is:

$$\delta Q = U(T_h - T_c)dA \qquad (1)$$

The heat transfer rate can also be written as the change in enthalpy of each fluid (with the correct sign) between the area A and A+dA:

* for the hot fluid ($dT_h<0$)

$$\delta Q = -\dot{m}_h c_{p,h} dT_h \qquad (2)$$

* for the cold fluid ($dT_c>0$)

$$\delta Q = \dot{m}_c c_{p,c} dT_c \qquad (3)$$

The notion of heat capacity can be introduced as:

$$C = \dot{m} c_p \qquad (4)$$

Equation (2) and (3) give:

$$\delta Q = -C_h dT_h = C_c dT_c \qquad (5)$$

Equations (1) and (5) give:

$$\frac{dT_h}{T_h - T_c} = -\frac{U}{C_h} dA \qquad (6)$$

$$\frac{dT_c}{T_h - T_c} = -\frac{U}{C_c} dA \qquad (7)$$

Subtracting equation (7) from (6):

$$\frac{d(T_h - T_c)}{T_h - T_c} = \left(\frac{1}{C_c} - \frac{1}{C_h}\right) UdA \qquad (8)$$

Considering the overall heat transfer coefficient U=constant, equation (8) can be integrated:

$$\ln\left(T_h - T_c\right) = \left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA + \ln B \qquad (9)$$

$$T_h - T_c = B\exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] \qquad (10)$$

The constant of integration, K is obtained from the boundary condition at the inlet:

at A=0, $T_h - T_c = T_{h1} - T_{c2}$ (11)

$$K = T_{h1} - T_{c2} \qquad (12)$$

Introducing equation (12) in (10) we have:

$$\frac{T_h - T_c}{T_{h1} - T_{c2}} = \exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] \qquad (13)$$

From equation (10) it can be observed that the temperature difference T_h-T_c is an exponential function of surface area A, and T_h-$T_c \to 0$ when $A \to 0$. The variation of the hot fluid temperature and that of the cold fluid temperature can be obtained separately. By multiplying equations (6) and (13):

$$\frac{dT_h}{T_{h1} - T_{c2}} = -\frac{U}{C_h}\exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right]dA \qquad (14)$$

Integrating:

$$\frac{T_h}{T_{h1} - T_{c1}} = -\frac{U}{C_h}\frac{\exp\left[-\left(\frac{1}{C_h} + \frac{1}{C_c}\right)UA\right]}{-\frac{C_h + C_c}{C_h C_c}U} + B \qquad (15)$$

$$\frac{T_h}{T_{h1} - T_{c2}} = \frac{C_c}{C_c - C_h}\exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] + B \qquad (16)$$

The constant of integration, B is obtained from the boundary condition:

at A=0, $T_h = T_{h1}$, and

$$B = \frac{T_{h1}}{T_{h1} - T_{c2}} - \frac{C_c}{C_c - C_h} \qquad (17)$$

From (16) and (17) we have:

$$\frac{T_h - T_{h1}}{T_{h1} - T_{c1}} = \frac{-C_c}{C_h + C_c}\left\{1 - \exp\left[-\left(\frac{1}{C_h} + \frac{1}{C_c}\right)UA\right]\right\} \qquad (18)$$

From equations (7) and (13) following the same procedure we obtain:

$$\frac{T_c - T_{c1}}{T_{h1} - T_{c1}} = \frac{C_h}{C_h + C_c}\left\{1 - e^{-\left(\frac{1}{C_h} + \frac{1}{C_c}\right)UA}\right\} \qquad (19)$$

Equation (10) shows that for $A \to \infty$, $T_h = T_c = T_\infty$.

The value of T_∞ can be calculated, for example, from equation (19):

$$T_\infty = T_{c1} + \frac{C_c}{C_h + C_c}\left(T_{h1} - T_{c1}\right) \tag{20}$$

$$T_\infty = \frac{C_h T_{h1} + C_c T_{c1}}{C_h + C_c} \tag{21}$$

Problem 2.3

Show that the variation of the hot and cold fluid temperature along a counterflow heat exchanger is given by

$$\frac{T_h - T_{h1}}{T_{h1} - T_{c2}} = \frac{C_c}{C_c - C_h}\left\{\exp\left[\left(\frac{1}{Cc} - \frac{1}{C_h}\right)UA\right] - 1\right\}$$

and

$$\frac{T_c - T_{c2}}{T_{h1} - T_{c2}} = \frac{C_h}{C_c - C_h}\left\{\exp\left[\left(\frac{1}{Cc} - \frac{1}{C_h}\right)UA\right] - 1\right\}$$

SOLUTION:

$$\frac{dT_h}{T_h - T_c} = -\frac{U}{C_h}dA \qquad (1)$$

$$\frac{dT_c}{T_h - T_c} = -\frac{U}{C_c}dA \qquad (2)$$

Subtracting equation (2) from (1):

$$\frac{d(T_h - T_c)}{T_h - T_c} = \left(\frac{1}{C_c} - \frac{1}{C_h}\right)UdA \qquad (3)$$

Integrating for constant values of U, C_c and C_h we have

$$\ln(T_h - T_c) = \left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA + \ln B$$

$$T_h - T_c = B\exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] \qquad (4)$$

where B the constant of integration results from the boundary condition:

at A=0, $T_h - T_c = T_{h1} - T_{c2}$

$$B = T_{h1} - T_{c2} \qquad (5)$$

Introducing equation (5) in (4):

$$\frac{T_h - T_c}{T_{h1} - T_{c2}} = \exp\left[\left(\frac{1}{C_c} - \frac{1}{C_h}\right)UA\right] \qquad (6)$$

Examining the evolution of T_h and T_c separately by multiplying equations (1) and (6), (2) and (6) respectively, we have:

$$\frac{dT_h}{T_{h1}-T_{c2}} = -\frac{U}{C_h}\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]dA \qquad (7.1)$$

$$\frac{dT_c}{T_{h1}-T_{c2}} = -\frac{U}{C_c}\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]dA \qquad (7.2)$$

Integrating:

$$\frac{T_h}{T_{h1}-T_{c2}} = -\frac{U}{C_h}\frac{\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]}{\left(\frac{1}{C_c}-\frac{1}{C_h}\right)U}+B$$

$$\frac{T_h}{T_{h1}-T_{c2}} = \frac{C_c}{C_c-C_h}\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]+B \qquad (8.1)$$

$$\frac{T_c}{T_{h1}-T_{c2}} = -\frac{U}{C_c}\frac{\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]}{\left(\frac{1}{C_c}-\frac{1}{C_h}\right)U}+B'$$

$$\frac{T_c}{T_{h1}-T_{c2}} = \frac{C_h}{C_c-C_h}\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]+B' \qquad (8.2)$$

For A=0, $T_h=T_{h1}$, $T_c = T_{c2}$ and:

$$\frac{T_{h1}}{T_{h1}-T_{c2}} = \frac{C_c}{C_c-C_h}+B$$

$$B = \frac{T_{h1}}{T_{h1}-T_{c2}} - \frac{C_c}{C_c-C_h} \qquad (9.1)$$

$$\frac{T_{c2}}{T_{h1}-T_{c2}} = \frac{C_h}{C_c-C_h}+B'$$

$$B = \frac{T_{c2}}{T_{h1}-T_{c2}} - \frac{C_h}{C_c-C_h} \qquad (9.2)$$

Substituting (9.1) in (8.1), (9.2) in (8.2), respectively:

$$\frac{T_h-T_{h1}}{T_{h1}-T_{c2}} = \frac{C_c}{C_c-C_h}\left\{\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]-1\right\} \qquad (10.1)$$

$$\frac{T_c-T_{c2}}{T_{h1}-T_{c2}} = \frac{C_h}{C_c-C_h}\left\{\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]-1\right\} \qquad (10.2)$$

Problem 2.4

From problem 2.3, show that for the case $C_h<C_c$, $\frac{d^2T_h}{dA^2}>0$ and $\frac{d^2T_c}{dA^2}>0$, and therefore temperature curves are convex and for the case $C_h>C_c$, $\frac{d^2T_h}{dA^2}<0$, and $\frac{d^2T_c}{dA^2}<0$, therefore, the temperature curves are concave (see Figure 2.6).

SOLUTION:

The hot fluid has a smaller heat capacity than the cold fluid, that is why it is the one who "commands the transfer"

Differentiating equation (10.1) in problem 2.3:

$$dT_h = d(T_h - T_c)$$

$$\frac{dT_h}{dA} = (T_{h1} - T_{c2})\left(-\frac{1}{C_h}\right)U\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]$$

$$\frac{d^2T_h}{dA^2} = (T_{h1} - T_{c2})\left(-\frac{1}{C_h}\right)\left(\frac{1}{C_c}-\frac{1}{C_h}\right)U^2\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]$$

$$\frac{d^2T_h}{dA^2} = \frac{(T_{h1} - T_{c2})(C_c - C_h)}{C_c C_h^{2}}U^2\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right] > 0 \qquad (1)$$

Similarly, from equation (10.2):

$$\frac{dT_c}{dA} = (T_{h1} - T_{c2})\left(-\frac{1}{C_c}\right)U\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right]$$

$$\frac{d^2T_c}{dA^2} = \frac{(T_{h1} - T_{c2})(C_c - C_h)}{C_c^{2} C_h}U^2\exp\left[\left(\frac{1}{C_c}-\frac{1}{C_h}\right)UA\right] > 0 \qquad (2)$$

Since, the second derivatives with respect to area of both T_h and T_c are positive as seen in equations (1) and (2), both the temperature curves are convex.

Problem 2.5

Show that when the heat capacities of hot and cold fluids are equal (C_c=C_h=C), the variation of the hot and cold fluid temperature along a counter flow heat exchanger are linear with the surface area as:

$$\frac{T_c - T_{c_2}}{T_{h_1} - T_{c_2}} = \frac{T_h - T_{h_1}}{T_{h_1} - T_{c_2}} = -\frac{UA}{C}$$

SOLUTION:

When the two fluids have the same heat capacity, from equation (6) in problem 2.3:

$$\delta Q = U(T_h - T_c)dA = -C_h dT_h \qquad (1)$$

In equation (10.2) in problem 2.3 when $C_c \rightarrow C_h$ we have:

$$\frac{T_c - T_{c2}}{T_{h1} - T_{c2}} = \lim \frac{C_h}{C_c - C_h}\left(e^{\frac{C_h - C_c}{C_c C_h}UA} - 1\right) = \lim\left(-C_h \frac{UA}{C_c C_h}\right) = -\frac{UA}{C_c} \qquad (2)$$

Similarly, from equation (10.1) in problem 2.3:

$$\frac{d(T_h - T_c)}{T_h - T_c} = -\frac{U}{C_c}dA, \qquad \text{When } C_c \rightarrow C_h \qquad (3)$$

But C_c=C_h=C and from (2) and (3):

$$\ln(T_h - T_c) = -\frac{UA}{C_h} + \ln D \qquad (4)$$

Problem 2.6

Assume that in a condenser, there will be no-subcooling and condensate leaves the condenser at saturation temperature, T_h. Show that variation of the coolant temperature along the condenser is given by

$$\frac{T_c - T_{c1}}{T_h - T_{c1}} = 1 - \exp\left[-\frac{UA}{C_c}\right]$$

SOLUTION:

The heat transferred along a surface element dA is:

$$\delta Q = U(T_h - T_c)dA = -C_h dT_h \qquad (1)$$

Because T_h = constant in a condenser, we can write:

$$dT_h = d(T_h - T_c) \qquad (2)$$

Using equations (1) and (2):

$$\frac{d(T_h - T_c)}{T_h - T_c} = -\frac{U}{C_c}dA\ , \qquad (3)$$

Integrating:

$$\ln(T_h - T_c) = -\frac{UA}{C_h} + \ln D$$

$$T_h - T_c = B\exp\left(-\frac{U}{C_c}A\right) \qquad (4)$$

The constant of integration, B can be calculated with the boundary condition:

$T_c = T_{c1}$, for A=0.

$$T_h - T_{c1} = B \qquad (5)$$

The temperature distribution for the cold fluid can be obtained by introducing (5) in (4) as:

$$T_h - T_c = (T_h - T_{c1})\exp\left[-\frac{UA}{C_c}\right]$$

$$\frac{T_c - T_{c1}}{T_h - T_{c1}} = 1 - \exp\left(-\frac{UA}{C_c}\right)$$

Problem 2.7

In a boiler (evaporator), the temperature of hot gases decreases from T_{h1} to T_{h2}, while boiling occurs at a constant temperature T_c. Obtain an expression, as in Problem 2.6, for the variation of hot fluid temperature with the surface area.

SOLUTION:

The rate of heat transfer δQ across the heat transfer area dA can be expressed as:

$$\delta Q = U(T_h - T_c)dA = -C_h dT_h \qquad (1)$$

In an evaporator T_c = constant and

$$dT_h = d(T_h - T_c) \qquad (2)$$

From equations (1) and (2):

$$\frac{d(T_h - T_c)}{T_h - T_c} = -\frac{U}{C_h}dA \qquad (3)$$

$$\ln(T_h - T_c) = -\frac{UA}{C_h} + \ln D$$

$$T_h - T_c = D\exp\left(-\frac{UA}{C_h}\right) \qquad (4)$$

The boundary condition at A=0 gives the value of the constant D:

at A=0 $\quad T_h = T_{h1}$

$$T_{h1} - T_c = D \qquad (5)$$

Introducing (5) in (4):

$$T_h - T_c = (T_{h1} - T_c)\exp\left(-\frac{U}{C_h}A\right) \qquad (6)$$

Rearranging:

$$1 - \frac{T_h - T_c}{T_{h1} - T_c} = 1 - \exp\left(-\frac{U}{C_h}A\right)$$

$$\frac{T_h - T_{h1}}{T_{h1} - T_c} = -\left[1 - \exp\left(-\frac{U}{C_h}A\right)\right] \qquad (7)$$

Problem 2.8

Show that Eq. (2.46) is also applicable for $C_h > C_c$, that is $C^* = C_c/C_h$.

SOLUTION:

From Eq. (2.26b)

$$T_{h2} - T_{c1} = (T_{h1} - T_{c2})\exp\left[UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\right] \tag{1}$$

For the case $C_h > C_c$, $C_c = C_{\min}$, $C_h = C_{\max}$,

$$T_{h2} - T_{c1} = (T_{h1} - T_{c2})\exp\left[\frac{UA}{C_{\min}}\left(1 - \frac{C_c}{C_h}\right)\right] \tag{2}$$

$$= (T_{h1} - T_{c2})\exp[NTU(1 - C^*)]$$

From heat balance equation

$$C_c(T_{c2} - T_{c1}) = C_h(T_{h1} - T_{h2}) \tag{3}$$

or

$$C^*(T_{c2} - T_{c1}) = (T_{h1} - T_{h2}) \tag{4}$$

The heat exchanger efficiency

$$\varepsilon = \frac{Q}{Q_{\max}} = \frac{C_{\min}(T_{c2} - T_{c1})}{C_{\min}(T_{h1} - T_{c1})} = \frac{T_{c2} - T_{c1}}{T_{h1} - T_{c1}}$$

$$= \frac{(T_{c2} - T_{c1})(1 - C^*)}{(T_{h1} - T_{c1})(1 - C^*)}$$

$$= \frac{T_{c2} - T_{c1} - C^*(T_{c2} - T_{c1})}{T_{h1} - T_{c1} - C^*(T_{h1} - T_{c1})} \tag{5}$$

$$= \frac{T_{c2} - T_{c1} - T_{h1} + T_{h2}}{T_{h2} - T_{c1} - C^*(T_{h1} - T_{c2}) - T_{h2} + T_{h1} - C^*T_{c2} + C^*T_{c1}}$$

$$= \frac{T_{h2} - T_{c1} - (T_{h1} - T_{c2})}{T_{h2} - T_{c1} - C^*(T_{h1} - T_{c2})}$$

or

$$\varepsilon = \frac{1 - \dfrac{T_{h1} - T_{c2}}{T_{h2} - T_{c1}}}{1 - C^* \dfrac{T_{h1} - T_{c2}}{T_{h2} - T_{c1}}} \qquad (6)$$

$$= \frac{1 - \exp[-NTU(1 - C^*)]}{1 - C^* \exp[-NTU(1 - C^*)]}$$

This proves that for $C_h > C_c$, Eq. (2.46) can also be derived from Eq. (2.16b).

Problem 2.9

Obtain the expression for exchanger heat transfer effectiveness, ε, for parallel flow given by Eq. (2.47).

SOLUTION:

From Eq. (2.26c)

$$T_{h2} - T_{c2} = (T_{h1} - T_{c1})\exp\left[-UA\left(\frac{1}{C_c} + \frac{1}{C_h}\right)\right] \quad (1)$$

Assume $C_h > C_c$, $C_c = C_{\min}$, $C_h = C_{\max}$,

$$T_{h2} - T_{c2} = (T_{h1} - T_{c1})\exp\left[-\frac{UA}{C_{\min}}\left(1 + \frac{C_c}{C_h}\right)\right] \quad (2)$$

$$= (T_{h1} - T_{c1})\exp[-NTU(1 + C^*)]$$

From heat balance equation

$$C_c(T_{c2} - T_{c1}) = C_h(T_{h1} - T_{h2}) \quad (3)$$

or

$$C^*(T_{c2} - T_{c1}) = (T_{h1} - T_{h2}) \quad (4)$$

The heat exchanger efficiency

$$\varepsilon = \frac{Q}{Q_{\max}} = \frac{C_{\min}(T_{c2} - T_{c1})}{C_{\min}(T_{h1} - T_{c1})} = \frac{T_{c2} - T_{c1}}{T_{h1} - T_{c1}}$$

$$= \frac{(T_{c2} - T_{c1})(1 + C^*)}{(T_{h1} - T_{c1})(1 + C^*)}$$

$$= \frac{(T_{c2} - T_{c1})\left(1 + \frac{T_{h1} - T_{h2}}{T_{c2} - T_{c1}}\right)}{(T_{h1} - T_{c1})(1 + C^*)} \quad (5)$$

$$= \frac{T_{c2} - T_{c1} + T_{h1} - T_{h2}}{(T_{h1} - T_{c1})(1 + C^*)}$$

$$= \frac{(T_{h1} - T_{c1}) - (T_{h2} - T_{c2})}{(T_{h1} - T_{c1})(1 + C^*)}$$

$$= \frac{1 - \exp[-NTU(1 + C^*)]}{1 + C^*}$$

This proves that for $C_h > C_c$, Eq. (2.47) can be derived from Eq. (2.16c). For case $C_h < C_c$, similar result can also be obtained.