Chapter 2 Solutions

Problem 2.1

Let $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, or in indicial notation,

$$v_i \hat{\mathbf{e}}_i = a_j \hat{\mathbf{e}}_j \times b_k \hat{\mathbf{e}}_k = \varepsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i$$

Using indicial notation, show that,

(a) $\mathbf{v} \cdot \mathbf{v} = a^2 b^2 \sin^2 \theta$,

(b) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$,

(c) $\mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0$.

Solution

(a) For the given vector, we have

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{v} &= \varepsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i \cdot \varepsilon_{pqs} a_q b_s \hat{\mathbf{e}}_p = \varepsilon_{ijk} a_j b_k \varepsilon_{pqs} a_q b_s \delta_{ip} = \varepsilon_{ijk} a_j b_k \varepsilon_{iqs} a_q b_s \\
&= (\delta_{jq}\delta_{ks} - \delta_{js}\delta_{kq}) a_j b_k a_q b_s = a_j a_j b_k b_k - a_j b_k a_k b_j \\
&= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{b}) = a^2 b^2 - (ab \cos\theta)^2 \\
&= a^2 b^2 \left(1 - \cos^2\theta\right) = a^2 b^2 \sin^2 \theta
\end{aligned}$$

(b) Again, we find

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{a} = (\varepsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i) \cdot a_q \hat{\mathbf{e}}_q = \varepsilon_{ijk} a_j b_k a_q \delta_{iq} = \varepsilon_{ijk} a_j b_k a_i = 0$$

This is zero by symmetry in i and j.

(c) This is

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = \mathbf{v} \cdot \mathbf{b} = (\varepsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i) \cdot b_q \hat{\mathbf{e}}_q = \varepsilon_{ijk} a_j b_k b_q \delta_{iq} = \varepsilon_{ijk} a_j b_k b_i = 0$$

Again, this is zero by symmetry in k and and i.

Problem 2.2

With respect to the triad of base vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (not necessarily unit vectors), the triad $\mathbf{u}^1$,$\mathbf{u}^2$, and $\mathbf{u}^3$ is said to be a *reciprocal basis* if $\mathbf{u}_i \cdot \mathbf{u}^j = \delta_{ij}$ $(i, j = 1, 2, 3)$. Show that to satisfy these conditions,

$$\mathbf{u}^1 = \frac{\mathbf{u}_2 \times \mathbf{u}_3}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}; \quad \mathbf{u}^2 = \frac{\mathbf{u}_3 \times \mathbf{u}_1}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}; \quad \mathbf{u}^3 = \frac{\mathbf{u}_1 \times \mathbf{u}_2}{[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]}$$

and determine the reciprocal basis for the specific base vectors

$$\begin{aligned}
\mathbf{u}_1 &= 2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 , \\
\mathbf{u}_2 &= 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3 , \\
\mathbf{u}_3 &= \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 .
\end{aligned}$$

Answer

$$\begin{aligned}\mathbf{u}^1 &= \tfrac{1}{5}\left(3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3\right)\\ \mathbf{u}^2 &= \tfrac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\right)\\ \mathbf{u}^3 &= \tfrac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3\right)\end{aligned}$$

Solution

For the bases, we have

$$\mathbf{u}_1\cdot\mathbf{u}^1 = \mathbf{u}_1\cdot\frac{\mathbf{u}_2\times\mathbf{u}_3}{[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]} = 1; \quad \mathbf{u}_2\cdot\mathbf{u}^2 = \mathbf{u}_2\cdot\frac{\mathbf{u}_3\times\mathbf{u}_1}{[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]} = 1; \quad \mathbf{u}_3\cdot\mathbf{u}^3 = \mathbf{u}_3\cdot\frac{\mathbf{u}_1\times\mathbf{u}_2}{[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]} = 1$$

since the triple scalar product is insensitive to the order of the operations. Now

$$\mathbf{u}_2\cdot\mathbf{u}^1 = \mathbf{u}_2\cdot\frac{\mathbf{u}_2\times\mathbf{u}_3}{[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]} = 0$$

since $\mathbf{u}_2\cdot\mathbf{u}_2\times\mathbf{u}_3 = 0$ from Pb 2.1. Similarly, $\mathbf{u}_3\cdot\mathbf{u}^1 = \mathbf{u}_1\cdot\mathbf{u}^2 = \mathbf{u}_3\cdot\mathbf{u}^2 = \mathbf{u}_1\cdot\mathbf{u}^3 = \mathbf{u}_2\cdot\mathbf{u}^3 = 0$.

For the given vectors, we have

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3] = \begin{vmatrix} 2 & 1 & 0\\ 0 & 2 & -1\\ 1 & 1 & 1\end{vmatrix} = 5$$

and

$$\mathbf{u}_2\times\mathbf{u}_3 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 0 & 2 & -1\\ 1 & 1 & 1\end{vmatrix} = 3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3; \quad \mathbf{u}^1 = \frac{1}{5}\left(3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3\right)$$

$$\mathbf{u}_3\times\mathbf{u}_1 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 1 & 1 & 1\\ 2 & 1 & 0\end{vmatrix} = -\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3; \quad \mathbf{u}^2 = \frac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3\right)$$

$$\mathbf{u}_1\times\mathbf{u}_2 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ 2 & 1 & 0\\ 0 & 2 & -1\end{vmatrix} = -\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3; \quad \mathbf{u}^3 = \frac{1}{5}\left(-\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3\right)$$

Problem 2.3

Let the position vector of an arbitrary point $P(x_1x_2x_3)$ be $\mathbf{x} = x_i\hat{\mathbf{e}}_i$, and let $\mathbf{b} = b_i\hat{\mathbf{e}}_i$ be a *constant vector*. Show that $(\mathbf{x}-\mathbf{b})\cdot\mathbf{x} = 0$ is the vector equation of a spherical surface having its center at $\mathbf{x} = \frac{1}{2}\mathbf{b}$ with a radius of $\frac{1}{2}b$.

Solution

For

$$\begin{aligned}(\mathbf{x}-\mathbf{b})\cdot\mathbf{x} &= (x_i\hat{\mathbf{e}}_i - b_i\hat{\mathbf{e}}_i)\cdot x_j\hat{\mathbf{e}}_j = (x_ix_j - b_ix_j)\,\delta_{ij} = x_ix_i - b_ix_i =\\ &= x_1^2 + x_2^2 + x_3^2 - b_1x_1 - b_2x_2 - b_3x_3 = 0\end{aligned}$$

Now

$$\left(x_1 - \frac{1}{2}b_1\right)^2 + \left(x_2 - \frac{1}{2}b_2\right)^2 + \left(x_3 - \frac{1}{2}b_3\right)^2 = \frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2\right) = \frac{1}{4}b^2$$

This is the equation of a sphere with the desired properties.

Problem 2.4

Using the notations $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$ and $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$ show that

(a) the tensor $\mathbf{A}$ having components A_{ij} can always be decomposed into a sum of its symmetric $A_{(ij)}$ and skew-symmetric $A_{[ij]}$ parts, respectively, by the decomposition,

$$A_{ij} = A_{(ij)} + A_{[ij]} \ ,$$

(b) the trace of $\mathbf{A}$ is expressed in terms of $A_{(ij)}$ by

$$A_{ii} = A_{(ii)} \ ,$$

(c) for arbitrary tensors $\mathbf{A}$ and $\mathbf{B}$,

$$A_{ij}B_{ij} = A_{(ij)}B_{(ij)} + A_{[ij]}B_{[ij]} \ .$$

Solution

(a) The components can be written as

$$A_{ij} = \left(\frac{A_{ij} + A_{ji}}{2}\right) + \left(\frac{A_{ij} - A_{ji}}{2}\right) = A_{(ij)} + A_{[ij]}$$

(b) The trace of $\mathbf{A}$ is

$$A_{(ii)} = \left(\frac{A_{ii} + A_{ii}}{2}\right) = A_{ii}$$

(c) For two arbitrary tensors, we have

$$\begin{aligned} A_{ij}B_{ij} &= \left(A_{(ij)} + A_{[ij]}\right)\left(B_{(ij)} + B_{[ij]}\right) = A_{(ij)}B_{(ij)} + A_{[ij]}B_{(ij)} + A_{(ij)}B_{[ij]} + A_{[ij]}B_{[ij]} \\ &= A_{(ij)}B_{(ij)} + A_{[ij]}B_{[ij]} \end{aligned}$$

since the product of a symmetric and skew-symmetric tensor is zero

$$\begin{aligned} A_{(ij)}B_{[ij]} &= \left(\frac{A_{ij} + A_{ji}}{2}\right)\left(\frac{B_{ij} - B_{ji}}{2}\right) = \frac{1}{4}\left(A_{ij}B_{ij} + A_{ji}B_{ij} - A_{ij}B_{ji} - A_{ji}B_{ji}\right) \\ &= \frac{1}{4}\left(A_{ij}B_{ij} + A_{ji}B_{ij} - A_{ji}B_{ij} - A_{ij}B_{ij}\right) = 0 \end{aligned}$$

We have changed the dummy indices on the last two terms.

Problem 2.5

Expand the following expressions involving Kronecker deltas, and simplify where possible.

$$\text{(a) } \delta_{ij}\delta_{ij}, \quad \text{(b) } \delta_{ij}\delta_{jk}\delta_{ki}, \quad \text{(c) } \delta_{ij}\delta_{jk}, \quad \text{(d) } \delta_{ij}A_{ik}$$

Answer

(a) 3, (b) 3, (c) δ_{ik}, (d) A_{jk}

Solution
(a) Contracting on i or j, we have

$$\delta_{ij}\delta_{ij} = \delta_{jj} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

(b) Contracting on k and then j gives

$$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3$$

(c) Contracting on j yields

$$\delta_{ij}\delta_{jk} = \delta_{ik}$$

(d) Contracting on i gives

$$\delta_{ij}A_{ik} = A_{jk}$$

Note: It may be helpful for beginning students to write out all terms.

Problem 2.6
If $a_i = \varepsilon_{ijk}b_jc_k$ and $b_i = \varepsilon_{ijk}g_jh_k$, substitute b_j into the expression for a_i to show that

$$a_i = g_kc_kh_i - h_kc_kg_i \ ,$$

or in symbolic notation, $\mathbf{a} = (\mathbf{c}\cdot\mathbf{g})\mathbf{h} - (\mathbf{c}\cdot\mathbf{h})\mathbf{g}$.

Solution
We begin by changing the dummy indices for $b_j = \varepsilon_{jmn}g_mh_n$ and

$$\begin{aligned} a_i &= \varepsilon_{ijk}b_jc_k = \varepsilon_{ijk}\varepsilon_{jmn}g_mh_nc_k = -\left(\varepsilon_{jik}\varepsilon_{jmn}g_mh_nc_k\right) = -\left(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}\right)g_mh_nc_k \\ &= -g_ih_kc_k + g_kh_ic_k = g_kc_kh_i - h_kc_kg_i \end{aligned}$$

where we have used the anti-symmetry of $\varepsilon_{ijk} = -\varepsilon_{jik}$ and the $\varepsilon - \delta$ identity. Symbolically $\mathbf{a} = (\mathbf{c}\cdot\mathbf{g})\mathbf{h} - (\mathbf{c}\cdot\mathbf{h})\mathbf{g}$

Problem 2.7
By summing on the repeated subscripts determine the simplest form of

$$(a)\ \varepsilon_{3jk}a_ja_k, \quad (b)\ \varepsilon_{ijk}\delta_{kj}, \quad (c)\ \varepsilon_{1jk}a_2T_{kj}, \quad (d)\ \varepsilon_{1jk}\delta_{3j}v_k\ .$$

Answer

(a) 0, (b) 0, (c) $a_2(T_{32} - T_{23})$, (d) $-v_2$

Solution
(a) Summing gives

$$\varepsilon_{3jk}a_ja_k = \varepsilon_{31k}a_1a_k + \varepsilon_{32k}a_2a_k = \varepsilon_{312}a_1a_2 + \varepsilon_{321}a_2a_1 = a_1a_2 - a_2a_1 = 0$$

(b)

$$\begin{aligned}\varepsilon_{ijk}\delta_{kj} &= \varepsilon_{ij1}\delta_{1j} + \varepsilon_{ij2}\delta_{2j} + \varepsilon_{ij3}\delta_{3j} \\ &= \varepsilon_{i21}\delta_{12} + \varepsilon_{i31}\delta_{13} + \varepsilon_{i12}\delta_{21} + \varepsilon_{i32}\delta_{23} + \varepsilon_{i13}\delta_{31} + \varepsilon_{i23}\delta_{32} = 0\end{aligned}$$

(c)

$$\begin{aligned}\varepsilon_{1jk}a_2T_{kj} &= \varepsilon_{12k}a_2T_{k2} + \varepsilon_{13k}a_2T_{k3} \\ &= \varepsilon_{123}a_2T_{32} + \varepsilon_{132}a_2T_{23} = a_2T_{32} - a_2T_{23} = a_2\left(T_{32} - T_{23}\right)\end{aligned}$$

(d)

$$\varepsilon_{1jk}\delta_{3j}v_k = \varepsilon_{12k}\delta_{32}v_k + \varepsilon_{13k}\delta_{33}v_k = 0 + \varepsilon_{132}\delta_{33}v_2 = -v_2$$

Problem 2.8

Consider the tensor $B_{ik} = \varepsilon_{ijk}v_j$.

(a) Show that B_{ik} is skew-symmetric.

(b) Let B_{ij} be skew-symmetric, and consider the vector defined by $v_i = \varepsilon_{ijk}B_{jk}$ (often called the *dual vector* of the tensor $\mathbf{B}$). Show that $B_{mq} = \frac{1}{2}\varepsilon_{mqi}v_i$.

Solution

(a) For a tensor to be skew-symmetric, one has $A_{ij} = -A_{ji}$. For the given tensor

$$B_{ik} = \varepsilon_{ijk}v_j = -\varepsilon_{kji}v_j = -B_{ki}$$

(b) For the dual vector of the tensor $\mathbf{B}$, we have

$$\begin{aligned}\varepsilon_{mqi}v_i &= \varepsilon_{mqi}\varepsilon_{ijk}B_{jk} = \left(\delta_{mj}\delta_{qk} - \delta_{mk}\delta_{qj}\right)B_{jk} = B_{mq} - B_{qm} = [B_{mq} - (-B_{mq})] \\ &= 2B_{mq}\end{aligned}$$

since $\mathbf{B}$ is skew-symmetric.

Problem 2.9

Use indicial notation to show that

$$A_{mi}\varepsilon_{mjk} + A_{mj}\varepsilon_{imk} + A_{mk}\varepsilon_{ijm} = A_{mm}\varepsilon_{ijk}$$

where $\boldsymbol{A}$ is any tensor and ε_{ijk} is the permutation symbol.

Solution

Multiply both sides by ε_{ijk} and simplify

$$\begin{aligned}A_{mm}\varepsilon_{ijk}\varepsilon_{ijk} = 6A_{mm} &= A_{mi}\varepsilon_{mjk}\varepsilon_{ijk} + A_{mj}\varepsilon_{imk}\varepsilon_{ijk} + A_{mk}\varepsilon_{ijm}\varepsilon_{ijk} \\ &= A_{mi}2\delta_{mi} + A_{mj}2\delta_{mj} + A_{mk}2\delta_{mk} = 6A_{mm}\end{aligned}$$

Problem 2.10

If $A_{ij} = \delta_{ij}B_{kk} + 3B_{ij}$, determine B_{kk} and using that solve for B_{ij} in terms of A_{ij} and its first invariant, A_{ii}.

Answer

$B_{kk} = \frac{1}{6}A_{kk}; \quad B_{ij} = \frac{1}{3}A_{ij} - \frac{1}{18}\delta_{ij}A_{kk}$

Solution
Taking the trace of A_{ij} gives

$$A_{ii} = \delta_{ii}B_{kk} + 3B_{ii} = 3B_{kk} + 3B_{ii} = 6B_{kk}$$

since i and k are dummy indices. This gives

$$B_{kk} = \frac{1}{6}A_{kk}$$

Substituting for B_{kk} and solving for B_{ij} gives

$$3B_{ij} = A_{ij} - \frac{1}{6}\delta_{ij}A_{kk} \quad \text{or} \quad B_{ij} = \frac{1}{3}A_{ij} - \frac{1}{18}\delta_{ij}A_{kk}$$

Problem 2.11
Show that the value of the quadratic form $T_{ij}x_ix_j$ is unchanged if T_{ij} is replaced by its symmetric part, $\frac{1}{2}(T_{ij} + T_{ji})$.

Solution
The quadratic form becomes

$$T_{ij}x_ix_j = \frac{1}{2}(T_{ij} + T_{ji})x_ix_j = \frac{1}{2}(T_{ij}x_ix_j + T_{ji}x_ix_j) = \frac{1}{2}(T_{ij}x_ix_j + T_{ij}x_jx_i) = T_{ij}x_ix_j$$

since i and j are dummy indices and multiplication commutes.

Problem 2.12
With the aid of Eq 2.7, show that any skew symmetric tensor $\boldsymbol{W}$ may be written in terms of an *axial vector* ω_i given by

$$\omega_i = -\frac{1}{2}\varepsilon_{ijk}w_{jk}$$

where w_{jk} are the components of $\boldsymbol{W}$.

Solution
Multiply by ε_{imn}

$$\begin{aligned} \varepsilon_{imn}\omega_i &= -\tfrac{1}{2}\varepsilon_{imn}\varepsilon_{ijk}w_{jk} \\ &= -\tfrac{1}{2}\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)w_{jk} \\ &= -\tfrac{1}{2}\left(w_{mn} - w_{nm}\right) = w_{nm}\ , \end{aligned}$$

or,

$$\varepsilon_{mni}\omega_i = w_{nm}$$

Problem 2.13
Show by direct expansion (or otherwise) that the box product $\lambda = \varepsilon_{ijk}a_ib_jc_k$ is equal to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

Thus, by substituting $\mathcal{A}_{1i}$ for a_i, $\mathcal{A}_{2j}$ for b_j and $\mathcal{A}_{3k}$ for c_k, derive Eq 2.42 in the form $\det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk}\mathcal{A}_{1i}\mathcal{A}_{2j}\mathcal{A}_{3k}$ where $\mathcal{A}_{ij}$ are the elements of $\boldsymbol{\mathcal{A}}$.

Solution
Direct expansion gives

$$\begin{aligned}\lambda &= \varepsilon_{ijk}a_ib_jc_k = \varepsilon_{1jk}a_1b_jc_k + \varepsilon_{2jk}a_2b_jc_k + \varepsilon_{3jk}a_3b_jc_k \\ &= \varepsilon_{12k}a_1b_2c_k + \varepsilon_{13k}a_1b_3c_k + \varepsilon_{21k}a_2b_1c_k + \varepsilon_{23k}a_2b_3c_k + \varepsilon_{31k}a_3b_1c_k + \varepsilon_{32k}a_3b_2c_k \\ &= \varepsilon_{123}a_1b_2c_3 + \varepsilon_{132}a_1b_3c_2 + \varepsilon_{213}a_2b_1c_3 + \varepsilon_{231}a_2b_3c_1 + \varepsilon_{312}a_3b_1c_2 + \varepsilon_{321}a_3b_2c_1 \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\end{aligned}$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 = \lambda$$

Using the suggested substitutions for a_i, b_i, c_i, we have $\mathcal{A}_3$

$$\begin{aligned}\lambda &= \varepsilon_{ijk}\mathcal{A}_{1i}\mathcal{A}_{2j}\mathcal{A}_{3k} = \varepsilon_{1jk}\mathcal{A}_{11}\mathcal{A}_{2j}\mathcal{A}_{3k} + \varepsilon_{2jk}\mathcal{A}_{12}\mathcal{A}_{2j}\mathcal{A}_{3k} + \varepsilon_{3jk}\mathcal{A}_{13}\mathcal{A}_{2j}\mathcal{A}_{3k} \\ &= \varepsilon_{12k}\mathcal{A}_{11}\mathcal{A}_{22}\mathcal{A}_{3k} + \varepsilon_{13k}\mathcal{A}_{11}\mathcal{A}_{23}\mathcal{A}_{3k} + \varepsilon_{21k}\mathcal{A}_{12}\mathcal{A}_{21}\mathcal{A}_{3k} + \varepsilon_{23k}\mathcal{A}_{12}\mathcal{A}_{23}\mathcal{A}_{3k} \\ &\quad + \varepsilon_{31k}\mathcal{A}_{13}\mathcal{A}_{21}\mathcal{A}_{3k} + \varepsilon_{32k}\mathcal{A}_{13}\mathcal{A}_{22}\mathcal{A}_{3k} \\ &= \varepsilon_{123}\mathcal{A}_{11}\mathcal{A}_{22}\mathcal{A}_{33} + \varepsilon_{132}\mathcal{A}_{11}\mathcal{A}_{23}\mathcal{A}_{32} + \varepsilon_{213}\mathcal{A}_{12}\mathcal{A}_{21}\mathcal{A}_{33} + \varepsilon_{231}\mathcal{A}_{12}\mathcal{A}_{23}\mathcal{A}_{31} \\ &\quad + \varepsilon_{312}\mathcal{A}_{13}\mathcal{A}_{21}\mathcal{A}_{32} + \varepsilon_{321}\mathcal{A}_{13}\mathcal{A}_{22}\mathcal{A}_{31} \\ &= \mathcal{A}_{11}\mathcal{A}_{22}\mathcal{A}_{33} - \mathcal{A}_{11}\mathcal{A}_{23}\mathcal{A}_{32} - \mathcal{A}_{12}\mathcal{A}_{21}\mathcal{A}_{33} + \mathcal{A}_{12}\mathcal{A}_{23}\mathcal{A}_{31} + \mathcal{A}_{13}\mathcal{A}_{21}\mathcal{A}_{32} - \mathcal{A}_{13}\mathcal{A}_{22}\mathcal{A}_{31}\end{aligned}$$

and

$$\begin{vmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{vmatrix} = \mathcal{A}_{11}\mathcal{A}_{22}\mathcal{A}_{33} - \mathcal{A}_{11}\mathcal{A}_{23}\mathcal{A}_{32} + \mathcal{A}_{12}\mathcal{A}_{23}\mathcal{A}_{31} - \mathcal{A}_{12}\mathcal{A}_{21}\mathcal{A}_{33} + \mathcal{A}_{13}\mathcal{A}_{21}\mathcal{A}_{32} - \mathcal{A}_{13}\mathcal{A}_{22}\mathcal{A}_{31} = \lambda$$

Problem 2.14
Starting with Eq 2.42 of the text in the form

$$\det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk}\mathcal{A}_{i1}\mathcal{A}_{j2}\mathcal{A}_{k3}$$

show that by an arbitrary number of interchanges of columns of $\mathcal{A}_{ij}$ we obtain

$$\varepsilon_{qmn}\det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk}\mathcal{A}_{iq}\mathcal{A}_{jm}\mathcal{A}_{kn}$$

which is Eq 2.43. Further, multiply this equation by the appropriate permutation symbol to derive the formula

$$6\det \boldsymbol{\mathcal{A}} = \varepsilon_{qmn}\varepsilon_{ijk}\mathcal{A}_{iq}\mathcal{A}_{jm}\mathcal{A}_{kn}\ .$$

Solution
Each row or column change introduces a minus sign. After an arbitrary number of row and column changes, we have

$$\varepsilon_{qmn}\det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk}\mathcal{A}_{iq}\mathcal{A}_{jm}\mathcal{A}_{kn}$$

Multiplying by ε_{qmn} gives

$$\begin{aligned}\varepsilon_{qmn}\varepsilon_{qmn}\det\boldsymbol{A} &= (\delta_{mm}\delta_{nn}-\delta_{mn}\delta_{nm})\det\boldsymbol{A} = (3\cdot 3-\delta_{nn})\det\boldsymbol{A} \\ &= (9-3)\det\boldsymbol{A} = \varepsilon_{qmn}\varepsilon_{ijk}A_{iq}A_{jm}A_{kn}\end{aligned}$$

from the $\varepsilon-\delta$ identity.

Problem 2.15

Let the determinant of the tensor A_{ij} be given by

$$\det\boldsymbol{A} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} .$$

Since the interchange of any two rows or any two columns causes a sign change in the value of the determinant, show that after an arbitrary number of row and column interchanges

$$\begin{vmatrix} A_{mq} & A_{mr} & A_{ms} \\ A_{nq} & A_{nr} & A_{ns} \\ A_{pq} & A_{pr} & A_{ps} \end{vmatrix} = \varepsilon_{mnp}\varepsilon_{qrs}\det\boldsymbol{A} .$$

Now let $A_{ij} = \delta_{ij}$ in the above determinant which results in $\det\boldsymbol{A} = 1$ and, upon expansion, yields

$$\varepsilon_{mnp}\varepsilon_{qrs} = \delta_{mq}(\delta_{nr}\delta_{ps}-\delta_{ns}\delta_{pr}) - \delta_{mr}(\delta_{nq}\delta_{ps}-\delta_{ns}\delta_{pq}) + \delta_{ms}(\delta_{nq}\delta_{pr}-\delta_{nr}\delta_{pq}) .$$

Thus, by setting $p = q$, establish Eq 2.7 in the form

$$\varepsilon_{mnq}\varepsilon_{qrs} = \delta_{mr}\delta_{ns} - \delta_{ms}\delta_{nr} .$$

Solution

Letting $A_{ij} = \delta_{ij}$ in the determinant gives

$$\begin{vmatrix} \delta_{mq} & \delta_{mr} & \delta_{ms} \\ \delta_{nq} & \delta_{nr} & \delta_{ns} \\ \delta_{pq} & \delta_{pr} & \delta_{ps} \end{vmatrix}$$

$$= \delta_{mq}\left(\delta_{nr}\delta_{ps}-\delta_{ns}\delta_{pr}\right) - \delta_{mr}\left(\delta_{nq}\delta_{ps}-\delta_{ns}\delta_{pq}\right) + \delta_{ms}\left(\delta_{nq}\delta_{pr}-\delta_{nr}\delta_{pq}\right)$$

and

$$\varepsilon_{mnp}\varepsilon_{qrs} = \delta_{mq}(\delta_{nr}\delta_{ps}-\delta_{ns}\delta_{pr}) - \delta_{mr}(\delta_{nq}\delta_{ps}-\delta_{ns}\delta_{pq}) + \delta_{ms}(\delta_{nq}\delta_{pr}-\delta_{nr}\delta_{pq})$$

since

$$\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = \varepsilon_{123}\varepsilon_{123}\det\boldsymbol{A}$$

Setting $p = q$ gives

$$\begin{vmatrix} \delta_{mp} & \delta_{mr} & \delta_{ms} \\ \delta_{np} & \delta_{nr} & \delta_{ns} \\ \delta_{pp} & \delta_{pr} & \delta_{ps} \end{vmatrix}$$

$$\begin{aligned}&= \delta_{mp}\left(\delta_{nr}\delta_{ps}-\delta_{ns}\delta_{pr}\right) - \delta_{mr}\left(\delta_{np}\delta_{ps}-\delta_{ns}\delta_{pp}\right) + \delta_{ms}\left(\delta_{np}\delta_{pr}-\delta_{nr}\delta_{pp}\right) \\ &= \delta_{nr}\delta_{ms} - \delta_{ns}\delta_{mr} - \delta_{mr}\left(\delta_{ns}-3\delta_{ns}\right) + \delta_{ms}\left(\delta_{nr}-3\delta_{nr}\right) \\ &= \delta_{nr}\delta_{ms} - \delta_{ns}\delta_{mr} + 2\delta_{mr}\delta_{ns} - 2\delta_{ms}\delta_{nr} = \delta_{mr}\delta_{ns} - \delta_{ms}\delta_{nr} \\ &= \varepsilon_{pmn}\varepsilon_{prs}\end{aligned}$$

Problem 2.16
Show that the square matrices

$$[\mathcal{B}_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\mathcal{C}_{ij}] = \begin{bmatrix} 5 & 2 \\ -12 & -5 \end{bmatrix}$$

are both square roots of the identity matrix.

Solution
The product of the matrix with itself should be the identity matrix for it to be a square root. Thus

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & 2 \\ -12 & -5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -12 & -5 \end{bmatrix} = \begin{bmatrix} 25-24 & 10-10 \\ -60+60 & -24+25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 2.17
Using the square matrices below, demonstrate

(a) that the transpose of the square of a matrix is equal to the square of its transpose (Eq 2.36 with $n = 2$),
(b) that $(\mathcal{A}\mathcal{B})^T = \mathcal{B}^T\mathcal{A}^T$ as was proven in Example 2.33

$$[\mathcal{A}_{ij}] = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 5 & 1 & 2 \end{bmatrix}, \quad [\mathcal{B}_{ij}] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix}.$$

Solution
(a) For the matrix $\mathcal{A}$, we have

$$[\mathcal{A}_{ij}]^2 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 1 & 5 \\ 20 & 8 & 16 \\ 25 & 4 & 13 \end{bmatrix}$$

and

$$[\mathcal{A}_{ij}^T]^2 = \begin{bmatrix} 3 & 0 & 5 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 5 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 20 & 25 \\ 1 & 8 & 4 \\ 5 & 16 & 13 \end{bmatrix}$$

This shows that $\left(\mathcal{A}^2\right)^T = \left(\mathcal{A}^T\right)^2$. Similarly for $\mathcal{B}$, we have

$$[\mathcal{B}_{ij}]^2 = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 9 & 19 \\ 26 & 10 & 27 \\ 16 & 12 & 13 \end{bmatrix}$$

and

$$[\mathcal{B}_{ij}^T]^2 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 26 & 16 \\ 9 & 10 & 12 \\ 19 & 27 & 13 \end{bmatrix}$$

(b) For $(\mathcal{AB})^T = \mathcal{B}^T\mathcal{A}^T$, we have

$$[\mathcal{A}_{ij}]\,[\mathcal{B}_{ij}] = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 5 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 6 \\ 20 & 4 & 22 \\ 15 & 17 & 16 \end{bmatrix}$$

and

$$\left[\mathcal{B}^T_{ij}\right]\left[\mathcal{A}^T_{ij}\right] = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 5 \\ 0 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 20 & 15 \\ 9 & 4 & 17 \\ 6 & 22 & 16 \end{bmatrix}$$

The result is demonstrated.

Problem 2.18

Let $\mathcal{A}$ be any orthogonal matrix, i.e., $\mathcal{A}\mathcal{A}^T = \mathcal{A}\mathcal{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Thus, by using the results in Examples 2.9 and 2.10, show that $\det \mathcal{A} = \pm 1$.

Solution

From Example 2.9

$$\det\left(\mathcal{A}\mathcal{A}^T\right) = \det \mathcal{A} \det \mathcal{A}^T$$

and from Example 2.10,

$$\det \mathcal{A} = \det \mathcal{A}^T$$

Then

$$\det\left(\mathcal{A}\mathcal{A}^T\right) = \det \mathcal{A} \det \mathcal{A}^T = \det \mathcal{A} \det \mathcal{A} = (\det \mathcal{A})^2 = \det \mathbf{I} = 1$$

and

$$(\det \mathcal{A}) = \pm 1$$

Problem 2.19

A tensor is called *isotropic* if its components have the same set of values in every Cartesian coordinate system at a point. Assume that $\mathbf{T}$ is an isotropic tensor of rank two with components t_{ij} relative to axes $Ox_1x_2x_3$. Let axes $Ox'_1x'_2x'_3$ be obtained with respect to $Ox_1x_2x_3$ by a righthand rotation of 120° about the axis along $\hat{\mathbf{n}} = (\hat{\mathbf{e}} + \hat{\mathbf{e}} + \hat{\mathbf{e}})/\sqrt{3}$. Show by the transformation between these axes that $t_{11} = t_{22} = t_{33}$, as well as other relationships. Further, let axes $Ox''_1x''_2x''_3$ be obtained with respect to $Ox_1x_2x_3$ by a right-hand rotation of 90° about x_3. Thus, show by the additional considerations of this transformation that if $\mathbf{T}$ is any isotropic tensor of second order, it can be written as $\lambda\mathbf{I}$ where λ is a scalar and $\mathbf{I}$ is the identity tensor.

Solution

For a 120° rotation about the axis $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/\sqrt{3}$, the transformation matrix is

$$[a_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$