

## Computational Fluid Mechanics and Heat Transfer

### Solutions Manual

#### Chapter 2

##### 2.1

The solution of Laplace's equation is

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh[n\pi(y-1)]$$

To verify that the coefficient  $A_n$  given in Example 2.1 is correct, we can first use the boundary condition  $T(x, 0) = T_0$ . Multiply this equation by  $\sin(n\pi x)$ , and integrate from 0 to 1:

$$\int_0^1 T_0 \sin(n\pi x) dx = \frac{T_0 [1 - (-1)^n]}{n\pi} = A_n \sinh(-n\pi) \frac{1}{2}$$

Using the trigonometry identity  $\sinh(-x) = -\sinh(x)$  the coefficient becomes

$$A_n = \frac{2T_0 [(-1)^n - 1]}{n\pi \sinh(n\pi)}$$

##### 2.2

For this problem,  $F(r, \theta) = r - r_b = 0$ , thus  $\nabla F = \mathbf{i}_r$  and the boundary condition is

$$u_r = \mathbf{V} \cdot \mathbf{i}_r = \nabla \phi \cdot \mathbf{i}_r = \frac{\partial \phi}{\partial r} = 0. \text{ Since } \phi = V_{\infty} r \cos \theta + K \cos \theta / r, \text{ we have } u_r = \cos \theta \left( V_{\infty} - \frac{K}{r^2} \right). \text{ The}$$

quantity in parenthesis must vanish on the cylinder ( $r = r_b$ ) so  $K = r_b^2 V_{\infty}$  and the required velocity boundary condition is satisfied.

### 2.3

Classical separation of variable provides the general term  $X(x)T(t)$ . Substituting into the wave equation  $y_{tt} = a^2 y_{xx}$  yields the following set of differential equations:

$$X'' + \alpha^2 X = 0 \quad T'' + \alpha^2 a^2 T = 0$$

The boundary and initial conditions are

$$X(0) = X(l) = 0 \quad T(0) = \sin\left(\frac{\pi x}{l}\right) \quad T'(t) = 0$$

This leads to a solution

$$y(x,t) = \sum A_n \sin\left(\frac{an\pi t}{l}\right) \cos\left(\frac{n\pi x}{l}\right)$$

In this case, only one term of the expansion is necessary to satisfy the specified initial displacement. Applying the boundary conditions eliminates all but the first term in the series.

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### 2.5

Applying the transformation to Equation 2.18a for the hyperbolic case results in the equation

$$-\frac{b^2 - 4ac}{a} \phi_{\xi\eta} + (e - d\lambda_1) \phi_{\xi} + (e - d\lambda_2) \phi_{\eta} + f \phi_{\eta} = \phi = g(\xi, \eta)$$


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### 2.6

Let  $\lambda_2 = \frac{b}{2a}$  and  $\lambda_1 = c$ . These selections provide transformed coordinates that are linearly

independent. The coefficient of the  $\phi_{\xi\eta}$  term is

$$a\lambda_1^2 - b\lambda_2 + c = -b^2 + 4ac = 0$$

and the cross derivative coefficient is

$$2a(\lambda_1\lambda_2) - b(\lambda_1 + \lambda_2) + 2c = -b^2 + 4ac = 0$$

and the correct form is obtained.

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## 2.7

The divergence theorem is  $\iint_D \nabla^2 u dA = \int_B \frac{\partial u}{\partial n} dl$ . Since the original equation is Laplace's equation on the domain  $D$ , the integral must vanish and substituting  $r = 1$  on the boundary yields

$$\int_B f(\theta) d\theta = \int_B \frac{\partial u}{\partial n}(1) d\theta = 0$$


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## 2.8

(a) For the equation  $y^2 u_{xx} - x^2 u_{yy} = 0$  we have  $a = y^2$ ,  $b = 0$ ,  $c = -x^2$ ,  $b^2 - 4ac = 4x^2 y^2$ .

The discriminant is positive so the equation is always hyperbolic except when  $x = 0$  and  $y = 0$ . For this isolated case, the equation is parabolic.

(b) Let  $\xi = x^2 + y^2$  and  $\eta = x^2 - y^2$ . The equation is transformed to

$$2(\xi^2 - \eta^2)u_{\xi\eta} - \eta u_\xi + \xi u_\eta = 0$$


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## 2.9

(a)  $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$  The discriminant is zero so the equation is parabolic.

(b)  $\xi = y + kx$ ,  $\eta = y - x$  assuming the second characteristic is a constant  $k \neq 1$ .

(c)  $2v_x - 4w_x + 2w_y + 3u = 0$

$$w_x - v_y = 0$$

Letting  $\mathbf{Z} = (v, w)$

$$[A]\mathbf{Z}_x + [C]\mathbf{Z}_y = [F]$$

where  $[A] = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$ ,  $[C] = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$  and  $[F] = \begin{bmatrix} -3u \\ 0 \end{bmatrix}$ .

(d)  $D = (4)^2 - 4(2)(2) = 0$  Therefore, the system of equations is parabolic.

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**2.10**

Classify the system of equations:

$$\frac{\partial u}{\partial t} + 8 \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + 2 \frac{\partial u}{\partial x} = 0$$

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 0, \quad d_1 = 8$$
$$a_2 = 0, \quad b_2 = 1, \quad c_2 = 2, \quad d_2 = 0$$

Since  $D > 0$ , the system of equations is hyperbolic.

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**2.11**

Answer: The equation is elliptic for all values of  $a \neq 0$ .

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**2.12**

Answer: hyperbolic

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**2.13**

Answer: elliptic, hyperbolic

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**2.18**

Answer: elliptic, hyperbolic

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### 2.19

(a)  $f(x) = \sin x, 0 \leq x \leq \pi$

Cosine series is  $f(x) = \frac{a_0}{2} + \sum A_n \cos(n\pi x)$  where  $n = 1 \rightarrow \infty$ . In this series, all basis functions  $[\cos(n\pi x)]$  are orthogonal to the function that is to be expanded. Thus, all of the Fourier coefficients vanish except  $a_0 = \frac{2}{\pi}$ . For the prescribed function, this is the best that can be done with the cosine series.

(b) In this case  $f(x) = \cos x$  is itself the cosine series and only one term of the Fourier series survives.

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### 2.20

(a)  $a = 1, b = 3, c = 2$

$$\frac{dy}{dx} = 1, 2$$

(b)  $a = 1, b = 12, c = 2$

$$\frac{dy}{dx} = -1$$

This is a parabolic equation and the other characteristic may be chosen with the restriction that the two are linearly independent.

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### 2.21

(a) Hyperbolic  $\lambda_1 = 2, \lambda_2 = 1$ . Let  $\xi = y - x, \eta = y + 2x$ . This transforms to  $u_{\xi\eta} = 0$ .

(b) Parabolic  $\lambda_1 = -1$ . Let  $\lambda_2 = 1, \xi = y - x, \eta = y + x$ . This transforms to  $u_{\xi\xi} = 0$ .

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**2.22**

(a) Answer:  $u_{\xi\xi} + u_{\eta\eta} = 0$

(b) Answer:  $u_{\xi\xi} + u_{\eta\eta} + \frac{u_{\xi}}{4} = 0$

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**2.23**

(a)  $\lambda_1 = -3$ . Let  $\lambda_2 = 1$ ,  $\xi = y - x$ ,  $\eta = y + 3x$ . After transforming we have

$16u_{\xi\xi} - u_{\xi} + 3u_{\eta} - e^{xy} = 0$  where  $x = \frac{\eta - \xi}{4}$ ,  $y = \frac{\eta + 3\xi}{4}$ .

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**2.24**

Answer:  $u(x, y) = -\sinh(y - \pi) \frac{\sin x}{\sinh \pi} - 2 \sinh(y - \pi) \frac{\sin 2x}{\sinh 2\pi}$

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**2.25**

Answer:  $u(x, y) = -\frac{\sin x}{\sinh \pi} - \frac{2 \sin 2x}{\sinh 2\pi}$

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**2.26**

Answer:  $u(x, y) = \sum_{n=1}^{\infty} A_n \sinh[n(y - \pi)] \sin(nx)$

where  $A_n = -\frac{2}{n \sinh(n\pi)} \left[ 2 \left( \pi^2 - \frac{5}{n^2} - \frac{12}{n^2 n^4} \right) + \frac{2 \cos(n\pi)}{n} \left( \frac{10\pi}{n^2} - \frac{24}{\pi n^4} \right) \right]$

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**2.27**

Answer:  $T(x, y) = e^{-4\pi^2 t} \sin(2\pi x)$

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